EXISTENCE OF TAUT FOLIATIONS ON SEIFERT FIBERED HOMOLOGY 3-SPHERES

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Abstract. This paper concerns the problem of existence of taut foliations among 3-manifolds. Since the contribution of David Gabai [11], we know that closed 3-manifolds with non-trivial second homology group admit a taut foliations. The essential part of this paper focuses on Seifert fibered homology 3-spheres. The result is quite different if they are integral or rational but non-integral homology 3-spheres. Concerning integral homology 3-spheres, we prove that all but the 3-sphere and the Poincaré 3-sphere admit a taut foliation. Concerning non-integral homology 3-spheres, we prove there are infinitely many which admit a taut foliation, and infinitely many without taut foliation. Moreover, we show that the geometries do not determine the existence of taut foliations on non-integral Seifert fibered homology 3-spheres.

Contents

1. Introduction

All 3-manifolds are considered compact, connected and orientable. Taut foliations provide deep information on 3-manifolds and their contribution in understanding the topology and geometry of 3-manifolds is still in progress. The first result came from S. P. Novikov [20] in 1965, who proved that a 3-manifold which admit a taut foliation has to be irreducible or $\mathbb{S}^2 \times \mathbb{S}^1$. Since then, we know [22]

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that such manifolds have $\mathbb{R}^3$ for universal cover, and that their fundamental group is infinite [20] and Gromov negatively curved when the manifold is also toroidal [6]. Recently, W. P. Thurston has exhibit an approach with taut foliations towards the geometrization.

In [11], D. Gabai proved that a closed 3-manifold with a non-trivial second homology group admits a taut foliation. A lot of great works then concern the existence of taut foliations, see for examples [2, 1, 3, 7, 23]. This paper seeks to answer the question for Seifert fibered 3-manifolds. In the following, a *non-integral homology 3-sphere* means a rational homology 3-sphere, which is not an integral homology 3-sphere. The results are quite different if they are integral homology 3-spheres, or non-integral homology 3-spheres.

Theorem 1.1 (Main Theorem 1)**.** *Let M be a Seifert fibered integral homology 3-sphere. Then M admits a taut analytic foliation if and only if M is neither homeomorphic to the 3-sphere nor the Poincaré sphere.*

Concerning non-integral homology 3-spheres, the non-existence is not isolated. Of course, the 3-sphere and lens spaces do not admit a taut foliation, but for any choice of the number of exceptional slopes, there exist infinitely many which admit a taut foliation, and infinitely many which do not.

Theorem 1.2 (Main Theorem 2)**.** *Let n be a positive integer greater than two. Let $\mathcal{S}_n$ be the set of Seifert fibered 3-manifolds whith n exceptional fibers, which are non-integral homology 3-spheres. For each n :*

(i) *There exist infinitely many Seifert fibered manifolds in $\mathcal{S}_n$ which admit a taut analytic foliation; and*
(ii) *There exist infinitely many Seifert fibered manifolds in $\mathcal{S}_n$ which do not admit a taut $\mathcal{C}^2$-foliation.*
(iii) *There exist infinitely many Seifert fibered manifolds in $\mathcal{S}_3$ which do not admit a taut $\mathcal{C}^0$-foliation.*

Actually, by considering the normalized Seifert invariant $(0; b_0, b_1/a_1, \dots, b_n/a_n)$ of a Seifert fibered homology 3-sphere, and assuming that $b_0 \neq -1$ (nor $1-n$), then b_0 determines wether M does or does not admit a taut $\mathcal{C}^2$-foliation, see Theorem 4.1, which collects results in [8, 14, 19]. Note that there is a fiber-preserving homeomorphism of M which switches $b_0 = 1-n$ to $b_0 = -1$. Therefore, the problem remains open only for $b_0 = -1$. We will prove (see Theorem 7.1) that even if the 3-manifolds all are equipped with $b_0 = -1$, Main Theorem 2 is still true. To prove the non-existence of taut $\mathcal{C}^2$-foliations, we first prove that a taut $\mathcal{C}^2$-foliation can be isotoped to a horizontal one, and then use a characterization of horizontal foliations for Seifert fibered homology 3-spheres (see bellow for more details : schedule of the paper). So, the following result play a key-rule in the proof.

Theorem 1.3. *Let M be a Seifert fibered rational homology 3-sphere. Let n be the number of exceptional fibers of M. If $n > 3$ (resp. $n = 3$) then any taut $\mathcal{C}^2$-foliation (resp. $\mathcal{C}^0$-foliation) of M can be isotoped to be a horizontal foliation.*

Moreover, we will show that the geometries do not determine the existence of taut foliations on Seifert fibered rational homology 3-spheres.

Theorem 1.4. *Let M be a Seifert fibered rational homology 3-sphere. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, then M does not admit a taut $\mathcal{C}^2$-foliation.*

Remark 1.5. *There exist infinitely many such manifolds (see Section 7) but the converse is not true as says Theorem 7.1 : we can give infinitely many such manifolds, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry (and with $b_0 = -1$) but no taut $\mathcal{C}^2$-foliation.*

Theorem 1.6. *Let M be a Seifert fibered integral homology 3-sphere. If M admits the $\widetilde{SL}_2(\mathbb{R})$-geometry, then M is neither homeomorphic to the 3-sphere nor the Poincaré sphere.*

In particular (Main Theorem 1) M admits a taut analytic foliation.

Schedule of the paper. We organize the paper as follows.

In Section 2, we recall basic definitions and notations on Seifert fibered 3-manifolds, taut or horizontal foliations and well known results.

Section 3 is devoted to the proof of Theorem 1.3, which is based on Proposition 3.2, which claims that a transversely oriented and taut foliation of a closed 3-manifold cannot contain a separating compact leaf. Then, a taut $\mathcal{C}^2$-foliation of a Seifert fibered homology 3-sphere cannot contain a compact leaf (see Corollary 3.5). Therefore, it can be isotoped to be horizontal (see Theorem 3.6), by collecting the works on foliations [1, 8, 15, 18, 20, 27] of M. Brittenham, D. Eisenbud, U. Hirsch, G. Levitt, S. Matsumoto, W. Neumann, S. P. Novikov and W. P. Thurston.

Since a horizontal foliation is clearly a taut foliation, an immediate consequence is that a Seifert fibered rational homology 3-sphere, M say, admits a taut $\mathcal{C}^2$-foliation if and only if M admits a horizontal foliation (Corollary 3.1). This corollary was also proved by combining results [9, 14, 16, 17, 19, 21] of Y. Eliashberg, M. Jankins, P. Lisca, G. Matić, R. Naimi, W. Neumann, P. Ozsváth, A. I. Stipsciz, Z. Szabó and W. P. Thurston (for more details, see the end of Section 3).

The goal of Section 4 is a characterization of Seifert fibered rational homology 3-spheres, which admit a taut $\mathcal{C}^2$-foliation. Since a taut $\mathcal{C}^2$-foliation can be isotoped to be horizontal, we use the characterization [14, 19] of M. Jankins, R. Naimi and W. Neumann for horizontal foliations (for more details, see Section 4). This characterization gives rise to criteria to be satisfied by the Seifert invatiants.

Section 5 concerns the geometries of homology 3-spheres. We will prove the following result.

Proposition 1.7. *Let M be a Seifert fibered rational homology 3-sphere, with n exceptional fibers. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, then the following statements all are satisfied.*

- (i) $n \leq 4$.
- (ii) *If $n = 4$ then M admits the $\mathcal{N}il$-geometry, and is a non-integral homology 3-sphere.*
- (iii) *If M is an integral homology 3-sphere, then M admits the $\mathbb{S}^3$-geometry and is either homeomorphic to the 3-sphere or to the Poincaré sphere.*

We may note that if $n = 2$ then M is a lens space (including $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^2$).

We combine Proposition 1.7 with the criteria given by the characterization of Section 4, to prove Theorem 1.4.

Section 6, 7 and 8 are devoted respectively to the proof of Theorem 1.4, Theorem 7.1 and Main Theorem 1.

To prove Theorem 7.1, we first exhibit infinite families of Seifert fibered non-integral homology spheres, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry (and $b_0 = 1$). Then, we prove that they do satisfy (or do not satisfy) the criteria of the characterization described in Section 4.

To prove Main Theorem 1, we need to study more deeply these criteria.

Perspectives.
By F. Waldhausen, [28] we know that an incompressible compact surface in a Seifert fibered 3-manifold (not necessarily a homology 3-sphere) can be isotoped to be either horizontal or vertical. This is clearly not the same for foliations.

A vertical leaf is homeomorphic to either a 2-cylinder ($\mathbb{S}^1 \times \mathbb{R}$) or a 2-torus ($\mathbb{S}^1 \times \mathbb{S}^1$). Therefore, taut foliations are not necessary isotopic to vertical ones; and vice-versa, vertical foliations are not necessary isotopic to taut foliations, e.g. cylinders which wrap around two tori in a turbulization way; for more details, see [4]. But clearly, horizontal foliations are taut.

By Theorem 3.6, a taut $\mathcal{C}^2$-foliation can be isotoped to a horizontal foliation, if there is no compact leaf.

We wonder if a taut $\mathcal{C}^0$-foliation, without compact leaf, of a Seifert fibered 3-manifold can be isotoped to be horizontal and so analytic. By [3], there exist manifolds which admit taut $\mathcal{C}^0$-foliation but not taut $\mathcal{C}^2$-foliation. Therefore, that seems impossible in general, but the question is still open for homology 3-spheres.

Question 1.8. *Let $\mathcal{F}$ be a taut $\mathcal{C}^0$-foliation, without compact leaf, of a Seifert fibered homology 3-sphere. Can $\mathcal{F}$ be isotoped to be horizontal ?*

M. Brittenham [1], answers the question when the base is $\mathbb{S}^2$ with 3 exceptional fibers, see Remark 3.7 for more details.

Gluing Seifert fibered 3-manifolds with boundary components along some of them (or all) give *graph manifolds*. We wonder if we can classify graph manifolds without taut foliations, with their Seifert fibered pieces and gluing homeomorphims.

Question 1.9. *Let M be a graph 3-manifold. What kind of obstructions are there for M not to admit a taut foliation ?*

2. Preliminaries

We may recall here, that all 3-manifolds are considered compact, connected and orientable. This section is devoted to recall basic definitions and notations on Seifert fibered 3-manifolds, taut or horizontal foliations and well known results.

Notations Let M be a 3-manifold. If M is an integral homology sphere, resp. a rational homology sphere, we say that M is a $\mathbb{Z}HS$, resp. a $\mathbb{Q}HS$. Clearly, a $\mathbb{Z}HS$ is a $\mathbb{Q}HS$. If M is a $\mathbb{Z}HS$, resp. a $\mathbb{Q}HS$, and a Seifert fibered 3-manifold, we say that M is *a $\mathbb{Z}HS$, resp. a $\mathbb{Q}HS$, Seifert fibered 3-manifold.*

Separating surfaces and non-separating surfaces. A properly embedded surface F in a 3-manifold M is said to be *a separating surface* if $M-F$ is not connected;

otherwise, F is said to be *a non-separating surface* in M. If F is a separating surface, we call *the sides of* F the connected components of $M - F$. Note that if M is a $\mathbb{Q}HS$ manifold, then M does not contain any non-separating surface.

A 3-manifold is said to be *reducible* if M contains an *essential* 2*-sphere*, i.e. a 2-sphere which does not bound any 3-ball in M. Then, either M is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, or M is a non-trivial connected sum. If M is not a reducible 3-manifold, we say that M is an *irreducible* 3-manifold. We may note that all Seifert fibered 3-manifolds but $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$ are irreducible 3-manifolds.

Seifert fibered 3-manifolds. We can find the first definition of Seifert fibered 3-manifolds, called *fibered spaces* by H. Seifert, in [26]. We first consider fibered solid tori.

The standard solid torus V is said to be *p/q-fibered*, if V is foliated by circles, such that the core is a leaf, and all the other leaves are circles isotopic to the (p, q)-torus knot (i.e. they run p times in the meridional direction and q times in the longitudinal direction) where $q \neq 0$. A solid torus W is *$\mathbb{S}^1$-fibered* if W is foliated by circles, such that there exists a homeomorphism between W and the p/q-fibered standard solid torus V, which preserves the leaves. We may say that W is a *p/q-fibered solid torus*.

A 3-manifold M is said to be *a Seifert fibered* 3-manifold, or *a Seifert fiber space* if M is a disjoint union of simple circles, called *the fibers*, such that the regular neighborhood of each fiber is a $\mathbb{S}^1$-fibered solid torus. Let W be a p/q-fibered solid torus. If $q = 1$, we say that its core is *a regular fiber*; otherwise we say that its core is *an exceptional fiber* and q is *the multiplicity* of the exceptional fiber.

By D. B. A. Epstein [10] this is equivalent to say that M is a $\mathbb{S}^1$-bundle over a 2-orbifold.

Seifert invariants. In [25], H. Seifert developped numerical invariants, which give a complete classification of Seifert fibered 3-manifolds. Let M be a closed Seifert manifold based on an orientable surface of genus g, with n exceptional fibers. Let $V_1, \dots, V_n$ be the solid tori, which are regular neighborhood of each exceptional fiber. We do not need to consider non-orientable base surface here. If we remove these solid tori, we obtain a trivial $\mathbb{S}^1$-bundle over a genus g compact surface, whose boundary is a union of 2-tori $T_1, \dots, T_n$; where $T_i = \partial V_i$, for $i \in \{1, \dots, n\}$. Gluing back $V_1, \dots, V_n$ consists to assign to each of them a slope b_i/a_i : we glue V_i along T_i, such that the slope b_i/a_i on V_i bounds a meridian disk of V_i. Formally, if f and s represent respectively a fiber and a section on T_i, then the boundary of the meridian disk of V_i is attached along the slope represented by $a_i[s] + b_i[f]$ in $H_1(T_i, \mathbb{Z})$.

Clearly, $a_i \geq 2$ is the multiplicity of the core of V_i, and b_i depends on the choice of a section. Removing the regular neighborhood of a regular fiber, we obtain an integer slope b_0. Then, $g, b_0, b_1/a_1, \dots, b_n/a_n$ completely describe M. We denote M by $M(g; b_0, b_1/a_1, \dots, b_n/a_n)$, which is called *the Seifert invariant*.

Seifert normalized invariant and convention. New sections are obtained by Dehn twistings along the fiber (along annuli or tori); therefore a new section does not change b_i modulo a_i. Thus, we can fix b_0 so that $0 < b_i < a_i$ for $i \in \{1, \dots, n\}$.

That gives rise to the *Seifert normalized invariant* : $M(g; b_0, b_1/a_1, \dots, b_n/a_n)$; i.e. $0 < b_i < a_i$ for $i \in \{1, \dots, n\}$.

H. Seifert [25] showed that $M(g; b_0, b_1/a_1, \ldots, b_n/a_n)$ is fiber-preserving homeomorphic to $-M(g; -n-b_0, 1-b_1/a_1, \ldots, 1-b_n/a_n)$ where $-M$ denotes M with the opposite orientation. In all the following, we denote by Φ this isomorphism. Therefore, we may assume that $b_0 < 0$ otherwise we switch for $-n-b_0$. For more details, see [25] or [3, 13].

Every $\mathbb{Q}HS$ Seifert fibered 3-manifold M is based on $\mathbb{S}^2$. Indeed, every non-separating curve on the base surface induces a non-separating torus in M; which cannot be in a $\mathbb{Q}HS$. Hence, the base surface of a $\mathbb{Q}HS$ Seifert fibered 3-manifold is a 2-sphere.

From now on, we denote for convenience such M by $M(-b_0, b_1/a_1, \ldots, b_n/a_n)$, where $b_0 > 0$ and $0 < b_i < a_i$ for $i \in \{1, \ldots, n\}$. We will write :

$$M = M(-b_0, b_1/a_1, \ldots, b_n/a_n).$$

Euler number. When M has a unique fibration, we denote by $e(M)$ *the Euler number of its fibration*. Note that few Seifert 3-manifolds (lens spaces and a finite number of others) do not have a unique fibration, see [13] for more details; all of them but lens spaces and $\mathbb{S}^3$, are not homology 3-spheres.

$$e(M) = -b_0 + \sum_{i=1}^{n} b_i/a_i.$$

Taut foliations. Let M be a 3-manifold and $\mathcal{F}$ a foliation of M. A simple closed curve γ (respectively, a properly embedded simple arc, when $\partial M \neq \emptyset$) is called *a transverse loop* (respectively a *transverse arc*) if γ is transverse to $\mathcal{F}$, i.e. γ is transverse to every leaf $F \in \mathcal{F}$, such that $\gamma \cap F \neq \emptyset$.

We say that a foliation $\mathcal{F}$ is *taut*, if for every leaf of F of $\mathcal{F}$, there exists a transverse loop, or a transverse arc if $\partial M \neq \emptyset$, γ say, such that $\gamma \cap F \neq \emptyset$.

We end this part by the famous theorem of Gabai [11] on the existence of taut foliations, which is stated here for closed 3-manifolds.

Theorem 2.1 (D. Gabai, [11]). *Let M be a closed 3-manifold. If $H_2(M;\mathbb{Q})$ is non-trivial then M admits a taut foliation.*

Horizontal and vertical foliations. Let M be a Seifert fibered 3-manifold and $\mathcal{F}$ a foliation of M. We say that $\mathcal{F}$ is *horizontal* if each $\mathbb{S}^1$-fiber is a transverse loop to $\mathcal{F}$. We say that $\mathcal{F}$ is *vertical* if each leaf of $\mathcal{F}$ is $\mathbb{S}^1$-fibered, i.e. a disjoint union of $\mathbb{S}^1$-fibers.

Note that only Seifert fibered 3-manifolds are concerned by horizontal or vertical foliations. Horizontal foliations are sometimes just called *transverse foliations* to underline the fact that horizontal foliations are transverse to the $\mathbb{S}^1$-fibers.

Clearly, horizontal foliations are taut, because any transverse fiber (meeting a leaf) is the required transverse loop; so we have the following result.

Lemma 2.2. *A horizontal foliation is taut.*

3. Horizontal and taut $\mathcal{C}^2$-foliations in Seifert fibered homology 3-spheres

This section is devoted to the proof of Theorem 1.3; then with Lemma 2.2, we obtain :

Corollary 3.1. *Let M be a $\mathbb{Q}HS$ Seifert fibered 3-manifold. Let n be the number of exceptional fibers of M. If $n > 3$ (resp. $n = 3$) then, M admits a horizontal foliation if and only if M admits a taut $\mathcal{C}^2$-foliation (resp. a $\mathcal{C}^0$-foliation).*

There exists an alternative proof (but not direct) of this corollary; see at the end of this section.

Proof of Theorem 1.3.
In the light of known results on foliations [1, 8, 15, 18, 20, 27] of M. Brittenham, D. Eisenbud, U. Hirsch, G. Levitt, S. Matsumoto, W. Neumann, S. P. Novikov and W. P. Thurston (where Theorem 3.6 is their collection) it is sufficient to see that any taut foliation on a $\mathbb{Q}$HS Seifert fibered 3-manifold, has no compact leaf. Then the result follows by Corollary 3.5, which claims that no leaf in a taut foliation of a $\mathbb{Q}HS$ can be compact. □

Corollary 3.5 is an immediate consequence of Proposition 3.2, which concerns all (compact, oriented and connected) closed 3-manifolds, and can be generalized with some boundary conditions to 3-manifolds with non-empty boundary, see [4].

Proposition 3.2. *A transversely oriented and taut foliation of a closed 3-manifold, cannot contain a compact separating leaf.*

For example, foliations of 3-manifolds which admit a Reeb's component are not taut.

A taut foliation $\mathcal{F}$ is said to be *transversely oriented* if there exists a one-dimensional oriented foliation $\mathcal{G}$ transverse to $\mathcal{F}$. This is equivalent to say that the normal vector field to the tangent planes to the leaves of $\mathcal{F}$ is continuous (and nowhere vanishes); which gives the following consequence.

Lemma 3.3. *Let $\mathcal{F}$ be a transversely oriented foliation. If F is a separating compact leaf then the continuous normal vector field to the tangent planes to F, has all his vectors pointing the same side of F.*

Actually, only this property is used in the proof of Proposition 3.2. We wonder if taut foliations are transversely oriented, and vice-versa. In fact, there exist taut foliations which are not transversely oriented, see [4] for more details. The inverse is easy to construct, e.g a Reeb's component. We may note that there exist also foliations without non-orientable compact leaves, which are neither taut nor transversely oriented.

Lemma 3.4. *Let $\mathcal{F}$ be a taut foliation. If $\mathcal{F}$ does not contain a non-orientable surface then $\mathcal{F}$ is a transversely oriented foliation.*

Proof. Let $\mathcal{F}$ be a foliation. If all the leaves are orientable then we can consider a well defined normal vector field to the tangent planes of $\mathcal{F}$. Moreover, if the foliation is taut then we can choose a well defined and continuous vector field (and nowhere vanishing). Therefore, $\mathcal{F}$ is a transversely oriented foliation. □

Since a $\mathbb{Q}HS$ cannot contain a non-orientable surface neither a non-separating surface, Proposition 3.2 gives immediately the following result.

Corollary 3.5. *A taut foliation of a $\mathbb{Q}HS$ cannot admit a compact leaf.*

Proof of Proposition 3.2
Let M be a closed 3-manifold. Let $\mathcal{F}$ be a taut foliation. We proceed by contradiction. So, we assume that $\mathcal{F}$ contains a compact separating leaf, F say. We will see that $\mathcal{F}$ cannot be taut; which is the required contradiction. We follow here a M. Brittenham's argument in his notes [2] concerning the non-tatness of a Reeb's component.

Since F is separating, $M - F$ contains two components. Let M_1 be the closure of one component, and $\mathcal{F}_1 = \mathcal{F} \cap M_1$. Let γ be a properly embedded arc in M_1. We will show that γ cannot be transverse to $\mathcal{F}_1$; therefore $\mathcal{F}_1$ cannot be taut and $\mathcal{F}$ neither.

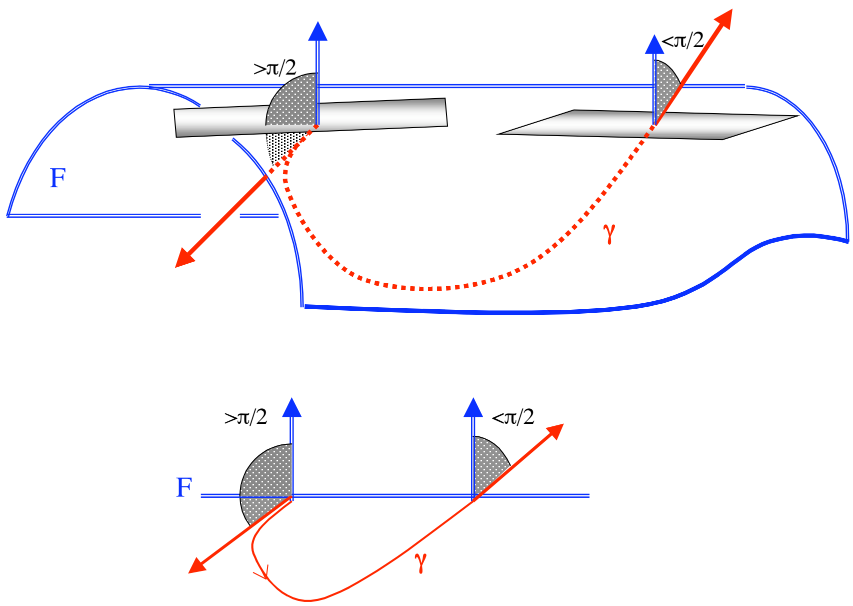

FIGURE 1.

Note that γ is a continuous map from $[0,1]$ to M_1. We consider the following map $h \ : [0,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ defined as follows :

$$\begin{array}{rccl} h & : & [0,1] & \to & [-\frac{\pi}{2}, \frac{\pi}{2}] \\ & & t & \mapsto & h(t) = \big(\widehat{\overrightarrow{v_{\gamma(t)}}, \overrightarrow{n_{\gamma(t)}}}\big) - \frac{\pi}{2} \end{array}$$

where $\big(\widehat{\overrightarrow{v_{\gamma(t)}}, \overrightarrow{n_{\gamma(t)}}}\big)$ is the non-oriented angle between the tangent vector to $\gamma(t)$, denoted by $\overrightarrow{v_{\gamma(t)}}$, and the normal vector to the tangent plane to the leaf of $\gamma(t)$, denoted by $\overrightarrow{n_{\gamma(t)}}$. Since M is a Rieman 3-manifold, h is trivially well defined. Since the vector fields are continuous, h is a continuous map.

We may note that $\big(\widehat{\overrightarrow{v_{\gamma(t)}}, \overrightarrow{n_{\gamma(t)}}}\big) \in [0, \pi]$ by definition. Note also that $h(t_0) = 0$ if and only if γ is tangent to the leaf at $\gamma(t_0)$. We will prove that there exists such a t_0; so γ is not transverse to $\mathcal{F}_1$.

Since h is a continuous map, by the Intermediate Value Theorem, it is sufficient to see that $h(0)h(1) < 0$; i.e. if $h(0) < 0$ then $h(1) > 0$ and vice-versa.
We may assume that $h(0)h(1) \neq 0$; otherwise $t_0 \in \{0, 1\}$. Since F is oriented, and $\mathcal{F}$ is transversely oriented, by Lemma 3.3, the normal vectors $\overrightarrow{n_{\gamma(0)}}$ and $\overrightarrow{n_{\gamma(1)}}$ both point outside M_1, or both point inside M_1 (and $n_{\gamma(t)}$ never vanishes).
On the other hand, since $\overrightarrow{v_{\gamma(t)}}$ never vanishes, and γ is properly embedded in M_1, the vectors $\overrightarrow{v_{\gamma(0)}}$ and $\overrightarrow{v_{\gamma(1)}}$ point in opposite sides of F; i.e. if $\overrightarrow{v_{\gamma(0)}}$ points inside M_1 then $\overrightarrow{v_{\gamma(1)}}$ points outside M_1 and vice-versa.
Therefore (see Figure 1) either :
$\overrightarrow{v_{\gamma(0)}}$ and $\overrightarrow{n_{\gamma(0)}}$ both point in the same side of M_1, and, $\overrightarrow{v_{\gamma(1)}}$ and $\overrightarrow{n_{\gamma(1)}}$ point opposite sides of M_1;
or
$\overrightarrow{v_{\gamma(1)}}$ and $\overrightarrow{n_{\gamma(1)}}$ both point in the same side of M_1, and, $\overrightarrow{v_{\gamma(0)}}$ and $\overrightarrow{n_{\gamma(0)}}$ point in opposite sides of M_1.

The former case means $h(0) < 0$ and $h(1) > 0$, while the later case means the inverse. Therefore $h(0)h(1) < 0$.

$\square$(Proposition 3.2)

Theorem 3.6 (collection of all [1, 8, 15, 18, 20, 27])**.** *Let M be a $\mathbb{Q}HS$ Seifert fibered 3-manifold, with n exeptional exceptional fibers (where $n \geq 3$). We assume that M admits a taut $\mathcal{C}^0$-foliation $\mathcal{F}$. Moreover, if $n > 3$, we suppose that $\mathcal{F}$ is a $\mathcal{C}^2$-foliation of M.*

If $\mathcal{F}$ does not have a compact leaf, then $\mathcal{F}$ can be isotoped to be a horizontal foliation.

Remark 3.7 (History on Theorem 3.6)**.** *This theorem has been proved for all Seifert 3-manifolds which are not trivial bundles over the 2-torus. This is a collection of results as follows.*
The case of circle bundles over orientable surface, which is not a 2-torus is due to W. P. Thurston in [27]*; it has been completed and extended to non-orientable base surface by G. Levitt in* [15]*.*
In [8]*, D. Eisenbud, U. Hirsch, and W. Neumann generalized it to Seifert fibered spaces, where the base surface is neither $\mathbb{S}^2$, nor the 2-torus with trivial circle bundle.*
Later, in [18]*, S. Matsumoto focused on the case when the base is $\mathbb{S}^2$ with stricly more than 3 exceptional fibers.*
Until there, the condition of $\mathcal{C}^r$-foliation is necessary, and implies a $\mathcal{C}^r$-isotopy, for each $r \geq 2$.
The last case (the base is $\mathbb{S}^2$ with 3 exceptional fibers) was solved by M. Brittenham [1]*, and the involved techniques are very different, so the author obtained a $\mathcal{C}^0$-isotopy from a $\mathcal{C}^0$-foliation.*
We may recall that when there are one or two exceptional fibers with base $\mathbb{S}^2$, there is no foliation without compact leaf by S. P. Novikov in [20]*.*

Alternative proof of Corollary 3.1
A proof of Corollary 3.1 has been obtained by combining the results of Y. Eliashberg and W. P. Thurston [9], M. Jankins and W. Neumann [14], P. Lisca and G. Matić [16], P. Lisca and A. I. Stipsicz [17], R. Naimi [19], and, P. Ozsváth and Z. Szabó [21], in the following way.

Theorem 3.8 ([9, 14, 16, 17, 19, 21]). *Let M be a rational Seifert fibered homology 3-sphere. The following statements are equivalent :*

(1) *M is an L-space;*
(2) *M does not carry a transverse contact structure;*
(3) *M does not admit a transverse foliation;*
(4) *M does not admit a taut foliation.*

This theorem is a formulation of [17, Theorem 1.1]. The proof is mainly organized as follows.

(1) $\Rightarrow$ (2) : P. Ozsváth and Z. Szabó [21];
(2) $\Rightarrow$ (3) : Y. Eliashberg and W. P. Thurston [9];
(1) $\Rightarrow$ (4) : P. Ozsváth and Z. Szabó [21];
(4) $\Rightarrow$ (3) : trivial;
(3) $\Rightarrow$ (2) : M. Jankins and W. Neumann [14], P. Lisca and G. Matić [16], and R. Naimi [19];
(2) $\Rightarrow$ (1) : P. Lipsca and A. I. Lipsicz [17].

To have (3) $\Rightarrow$ (4), we need to follow : (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4).

We may underline that the considered taut foliations are actually $\mathcal{C}^2$-foliations, because using contact structure (see [9]). Note that there exists a taut $\mathcal{C}^0$-foliation which is not a taut $\mathcal{C}^2$-foliation [3].

4. Characterization of taut $\mathcal{C}^2$-foliations in Seifert fibered homology 3-spheres

The goal of this section is to give a characterization of the existence of a taut $\mathcal{C}^2$-foliation in a $\mathbb{Q}$HS Seifert fibered 3-manifold. For this, we define the following *Property* $(*)$.

$$\textit{Property } (*) \begin{cases} (i) & \dfrac{b_1}{a_1} < \dfrac{m-\alpha}{m} \\[2ex] (ii) & \dfrac{b_2}{a_2} < \dfrac{\alpha}{m} \\[2ex] (iii) & \dfrac{b_i}{a_i} < \dfrac{1}{m}, \text{ for } i \in \{3, \dots, n\} \end{cases}$$

We say that *m and α satisfy Property $(*)$ for $b_1/a_1, b_2/a_2, \dots, b_n/a_n$*, if all the following statements are satisfied :

- m and α are two positive integers such that $\alpha < m$;
- $n \geq 3$ is an integer;
- a_i and b_j are positive integers for all $(i, j) \in \{1, \dots, n\}^2$, such that :
$$b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n;$$
- $(i), (ii)$ and (iii) of Property $(*)$ all are satisfied.

When there is no confusion for the b_i/a_i's, we say for short that (m, α) *satisfies Property* $(*)$, or that *the integers α and m satisfy Property* $(*)$.

For convenience, in the following, we denote by $(i), (ii)$ and (iii) respectively, the inequalities $(i), (ii)$ and (iii) of Property $(*)$ above.

Let M be a Seifert fibered 3-manifold. In the following, we use the previous notations (see Section 2) of Seifert normalized invariant :

$$M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$$

where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\} \times \{0,\ldots,n\}$, such that $0 < b_i < a_i$. Note that the notations $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$ suppose that M contains exactly n exceptional fibers :

$$a_i \geq 2, \text{ for all } i \in \{1, 2, \ldots, n\}.$$

If $b_0 \notin \{1, n-1\}$ then the existence of a taut $\mathcal{C}^2$-foliation depends uniquely of b_0, as suggests the following theorem.

Theorem 4.1 ([8, 14, 19])**.** *Let n be an integer and M be a Seifert manifold based on $\mathbb{S}^2$.*
We assume that $n \geq 3$ and that $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\} \times \{0,\ldots,n\}$. Then, all the following statements are satisfied.

1. *If $2 \leq b_0 \leq n-2$ then M admits a horizontal foliation.*
2. *If M admits a horizontal foliation then $1 \leq b_0 \leq n-1$.*
3. *If M admits a horizontal $\mathcal{C}^0$-foliation, then M admits a horizontal analytic foliation.*

Corollary 4.2. *Let n be an integer and M be a Seifert manifold based on $\mathbb{S}^2$.*
We assume that $n \geq 3$ and that $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\} \times \{0,\ldots,n\}$, and $b_0 \notin \{1, n-1\}$.
Then M admits an analytic horizontal foliation if and only if $2 \leq b_0 \leq n-2$.

Therefore, the problem falls on $b_0 = 1$; we may recall here (see Section 2) :

$$M(-1, b_1/a_1, \ldots, b_n/a_n) \cong -M(-(n-1), 1 - b_1/a_1, \ldots, 1 - b_n/a_n).$$

The following theorem is a consequence of Corollary 3.1 and the characterization of the existence of horizontal foliations in Seifert-fibered spaces based on $\mathbb{S}^2$, whose formulation can be found in [3, Proposition 6].

Theorem 4.3. *Let $n > 2$ be an integer and $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$ be a $\mathbb{Q}HS$ Seifert fibered 3-manifold; where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\}^2$.*

Assume that $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$.

If $n > 3$ (resp. $n = 3$) then, M admits a taut $\mathcal{C}^2$-foliation (resp. a taut $\mathcal{C}^0$-foliation) if and only if there exist two positive integers m and α such that (m, α) satisfies Property $(*)$.

We may recall that $\mathcal{P}$ denotes the Poincaré $\mathbb{Z}$HS, i.e. $\mathcal{P} = M(-1, 1/2, 1/3, 1/5)$. Note that Theorem 4.3 implies that $\mathcal{P}$ cannot admit a taut foliation, but this fact was already known by S. P. Novikov's Theorem, see [20] (because its π_1 is finite). Note also that if $n \in \{1, 2\}$ then M has to be $\mathbb{S}^3$ or a Lens space, which cannot admit a taut foliation.
Theorem 4.3 has the following corollaries, which will be useful for the next sections.

Corollary 4.4. *Let n be an integer and M be a $\mathbb{Q}HS$ Seifert fibered 3-manifold.*
We assume that $n \geq 3$ and that $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\}^2$.

We order the rational coefficients b_i/a_i such that : $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$. If for all $i \in \{1,\dots,n\}$, $\dfrac{b_i}{a_i} < \dfrac{1}{2}$ then M admits a taut $\mathcal{C}^2$-foliation.

Proof. With the notations and assumptions of the theorem, if $b_i/a_i < 1/2$, for all $i \in \{1,\dots,n\}$, then Property $(*)$ is satisfied, by choosing $m = 2$ and $\alpha = 1$. □

Corollary 4.5. *Let n be an integer and M be a $\mathbb{Q}HS$ Seifert fibered 3-manifold. We assume that $n \geq 3$ and that $M = M(-1, b_1/a_1, \dots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i,j) \in \{1,\dots,n\}^2$.*

We order the rational coefficients b_i/a_i such that : $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$.

If M admits a taut $\mathcal{C}^2$-foliation and $\dfrac{b_1}{a_1} \geq 1/2$, then the two following properties are both satisfied.

(1) $\dfrac{b_i}{a_i} < \dfrac{1}{2}$, *for all $i \geq 2$..*

(2) $\dfrac{b_n}{a_n} < \dfrac{1}{3}$. *In particular, $a_n \geq 4$.*

Proof. With the notations and assumptions of Theorem 4.3, if M admits a taut $\mathcal{C}^2$-foliation then we can find positive integers m, α such that $\alpha < m$ and Property $(*)$ is satisfied.

First, note that if $m = 2$ then $\alpha = 1$ and $b_1/a_1 < 1/2$, which is a contradiction to the hypothesis. Thus, $m \geq 3$.

Now, if $\dfrac{m-\alpha}{m} > \dfrac{1}{2}$ then $\dfrac{\alpha}{m} < \dfrac{1}{2}$, hence Property $(*)$ implies $\dfrac{b_i}{a_i} < \dfrac{1}{2}$ for $i \in \{2,\dots,n\}$ which proves (1).

Finally, assume that $\dfrac{b_1}{a_1} \geq \dfrac{1}{2} \geq \dfrac{b_2}{a_2} \geq \dfrac{b_3}{a_3} \geq \dfrac{1}{3}$. Then $b_3/a_3 \geq 1/m$ for all $m \geq 3$, so (iii) of $(*)$ cannot be satisfied. □

5. Geometries of Seifert fibered homology 3-spheres

The goal of this section is to recall general results on the geometries of Seifert fibered homology 3-spheres, and prove Proposition 1.7.

Let n be a positive integer and $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ be a $\mathbb{Q}$HS Seifert fibered 3-manifold. Recall that $e(M)$ denotes the Euler number of M, see Section 2. The following lemma is a well known result, see [5] for more details.

Lemma 5.1. *Let $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ be a Seifert fibered 3-manifold. Then :*

(1) *M is a $\mathbb{Z}HS$ if and only if $a_1 a_2 \dots a_n e(M) = \varepsilon$, where $\varepsilon \in \{-1,+1\}$;*
(2) *M is a $\mathbb{Q}HS$ if and only if $e(M) \neq 0$.*

Remark 5.2. *Note that* (1) *implies that the a_i's are pairwise relatively prime integers, therefore they are different.*

Then, we define the rational number χ_M as follows.

$$\chi_M = 2 - \sum_{i=1}^{n}(1 - \frac{1}{a_i}) = 2 - n + \sum_{i=1}^{n} \frac{1}{a_i}.$$

We have the following well known result (which can be found in [24] for example).

Proposition 5.3. *Let n be a positive integer and $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ be a $\mathbb{Q}HS$ Seifert fibered 3-manifold, then the following properties all are satisfied.*

(i) $\chi_M > 0 \Leftrightarrow M$ *admits the $\mathbb{S}^3$-geometry.*
(ii) $\chi_M < 0 \Leftrightarrow M$ *admits the $\widetilde{SL}_2(\mathbb{R})$-geometry.*
(iii) $\chi_M = 0 \Leftrightarrow M$ *admits the $\mathcal{N}$il-geometry.*

Proposition 5.4. *Let n be a positive integer and $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ be a $\mathbb{Q}HS$ Seifert fibered 3-manifold. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry then $n \leq 4$.*

Furthermore, if $n = 4$ then $M = M(-b_0, 1/2, 1/2, 1/2, 1/2)$ with $b_0 \neq 2$; so M admits the $\mathcal{N}$il-geometry and is a non-integral $\mathbb{Q}HS$.

Proof. Let n be a positive integer and $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ be a $\mathbb{Q}HS$. Assume that M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry. Then, by Proposition 5.3, $\chi_M \geq 0$. Therefore, $n - 2 \leq \displaystyle\sum_{i=1}^{n} \frac{1}{a_i}$.

Since, $a_i \geq 2$ for all $i \in \{1, \dots, n\}$, $n - 2 \leq \displaystyle\sum_{i=1}^{n} \frac{1}{a_i} \leq n/2 \Rightarrow n \leq 4$.

Now, assume first that $n = 4$. Then, $\displaystyle\sum_{i=1}^{4} \frac{1}{a_i} \geq 2$. On the other hand, $a_i \geq 2$ for all $i \in \{1, \dots, 4\}$, then $\displaystyle\sum_{i=1}^{4} \frac{1}{a_i} \leq 2$, and if one $a_i > 2$ then $\displaystyle\sum_{i=1}^{4} \frac{1}{a_i} < 2$.

Therefore, $a_i = 2$ for all $i \in \{1, \dots, 4\}$. Thus, $\chi_M = 0$ which means that M admits the $\mathcal{N}$il-geometry. Moreover, Lemma 5.1 (2) implies that $b_0 \neq 2$. Note that such M cannot be a $\mathbb{Z}HS$, by Remark 5.2. □

Corollary 5.5. *Let M be a $\mathbb{Z}HS$ Seifert fibered 3-manifold. Then, M has the $\widetilde{SL}_2(\mathbb{R})$-geometry or the $\mathbb{S}^3$-geometry.*

Furthermore, if M has the $\mathbb{S}^3$-geometry, then M is either homeomorphic to $\mathbb{S}^3$ or to the Poincaré sphere $\mathcal{P}$.

Proof. Let M be a $\mathbb{Z}HS$ Seifert fibered 3-manifold. Assume that M does not have the $\widetilde{SL}_2(\mathbb{R})$-geometry. Note that if $n \leq 2$ then M has to be homeomorphic to $\mathbb{S}^3$. By Proposition 5.4, we may assume that $n = 3$ and that $a_3 > a_2 > a_1 \geq 2$ (by remark 5.2).

Since $\chi_M \geq 0$, $\displaystyle\sum_{i=1}^{3} \frac{1}{a_i} \geq 1$. If $a_1 \geq 3$, then $\displaystyle\sum_{i=1}^{3} \frac{1}{a_i} \leq 1/3 + 1/4 + 1/5 < 1$, which is a contradiction. Then $a_1 = 2$. If $a_2 \neq 3$ then $a_2 \geq 5$ by remark 5.2. Hence, $\displaystyle\sum_{i=1}^{3} \frac{1}{a_i} \leq 1/2 + 1/5 + 1/7 < 1$, which is a contradiction. Therefore, $a_1 = 2$ and $a_2 = 3$. Similarly $a_3 = 5$. Since $n = 3$ and $(a_1, a_2, a_3) = (2, 3, 5)$, M has to be homeomorphic to the Poincaré sphere, which satisfies $\chi_M > 0$, so $\mathcal{P}$ has the $\mathbb{S}^3$-geometry. □

To end this section, we simply note that Proposition 5.4 together with Corollary 5.5 clearly imply Proposition 1.7.

6. Proof of Theorem 1.4

We keep the previous notations. Let n be a positive integer and M be a $\mathbb{Q}$HS Seifert fibered 3-manifold, with n exceptional fibers : $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$. Assume that M does not admit the $\widetilde{SL_2}(\mathbb{R})$-geometry. We make the proof by contradiction. Suppose that M admits a taut $\mathcal{C}^2$-foliation. We may recall that if $n \in \{1,2\}$, then M has a finite π_1, hence M cannot admit a taut $\mathcal{C}^2$-foliation. Therefore, by Proposition 5.4, we have $n \in \{3,4\}$.
Assume that $n = 4$. By Theorem 4.1 and Theorem 1.3, since M admits a taut $\mathcal{C}^2$-foliation, $b_0 \in \{1,2,3\}$. Moreover the cases $b_0 = 1$ and $b_0 = 3$ are equivalent (see the fiber-preserving homeomorphism Φ in Section 2).

On the other hand, Proposition 5.4 implies that $M = M(-b_0, 1/2, 1/2, 1/2, 1/2)$ with $b_0 \neq 2$; and Corollary 4.5 (1) implies that $b_0 \neq 1$.

Therefore, we may assume that $n = 3$. Similarly $b_0 \in \{1,2\}$ and $b_0 = 1$ and $b_0 = 2$ are equivalent cases, by considering the fiber-preserving homeomorphism Φ.

So, we may assume that $b_0 = 1$. Let $M = M(-1, b_1/a_1, b_2/a_2, b_3/a_3)$,

Since M is a $\mathbb{Q}$HS Seifert fibered 3-manifold, which does not admit the $\widetilde{SL_2}(\mathbb{R})$-geometry, Proposition 5.3 and Lemma 5.1(2) give respectively :

$$\begin{cases} (\mathrm{I1}) \quad \displaystyle\sum_{i=1}^{3} \frac{1}{a_i} \geq 1 \\ \text{and} \\ (\mathrm{I2}) \quad \displaystyle\sum_{i=1}^{3} \frac{b_i}{a_i} \neq 1 \end{cases}$$

By Corollary 4.5, we order the coefficients : $b_1/a_1 \geq b_2/a_2 \geq b_3/a_3$.

Let $a_{i_0} = \min(a_1, a_2, a_3)$. By (I1), $a_{i_0} \in \{2,3\}$.

First, we prove that a_{i_0} cannot be 3. We make the proof by contradiction. Assume that $a_{i_0} = 3$, then (I1) implies that $a_i = 3$ for all $i \in \{1,2,3\}$. Now, for all $i \in \{1,2,3\}$, $b_i < a_i$ so $b_i \leq 2$. If there exists $i \in \{1,2,3\}$ such that $b_i = 2$ then $b_i/a_i = 2/3 > 1/2$. But for $j \neq i$, $b_j/a_j \geq 1/3$, which is a contradiction to Corollary 4.5 (2). Therefore, $b_i/a_i = 1/3$, for all $i \in \{1,2,3\}$, which contradicts (I2).

Hence, we may assume that $a_{i_0} = 2$. Then :

$$b_{i_0}/a_{i_0} = 1/2.$$

By Corollary 4.5 (1), $b_{i_0}/a_{i_0} = b_1/a_1$.

Then Corollary 4.5 (1) and (2) imply respectively that $a_3 \geq 4$ and $a_2 \geq 3$.

Now (I1) implies that $\{a_1, a_2, a_3\}$ is one of the following sets :

$$\{2,3,4\}, \{2,3,5\}, \{2,3,6\} \text{ or } \{2,4,4\}.$$

We distinguish the cases $a_2 = 3$ and $a_2 = 4$.

Case 1 : $a_2 = 3$.
Then Corollary 4.5 (1) implies that :

$$b_2/a_2 = 1/3.$$

Now, by Theorem 4.3, there exist positive integers α and m which satisfy Property $(*)$. Now Corollary 4.5 (2) implies that : $\dfrac{b_3}{a_3} \in \left\{\dfrac{1}{4}, \dfrac{1}{5}, \dfrac{1}{6}\right\}$. Hence, by $(*)(iii)$, $m \leq 5$.

Since $b_1/a_1 = 1/2$, $m > 2$.

If $m = 3$ then $\alpha \in \{1, 2\}$, but in both cases $(*)(i)$ or $(*)(ii)$ cannot be satisfied. Similarly, if $m = 4$ then $\alpha \in \{1, 2, 3\}$, but in all cases $(*)(i)$ or $(*)(ii)$ cannot be satisfied.

If $m = 5$ then $a_3 = 6$ and $b_3 = 1$; otherwise $(*)(iii)$ cannot be satisfied.

Thus, $b_1/a_1 + b_2/a_2 + b_3/a_3 = 1/2 + 1/3 + 1/6 = 1$, which is in contradiction to $(I2)$, i.e. M cannot be a $\mathbb{Q}HS$.

Case 2 : $a_2 = 4$.

Then $a_2 = a_3 = 4$. Therefore Corollary 4.5 (1) implies that $\dfrac{b_2}{a_2} = \dfrac{b_3}{a_3} = \dfrac{1}{4}$. Therefore $(I2)$ is not satisfied, which is the final contradiction.

This ends the proof of Theorem 1.4.

7. Proof of Main Theorem 2

Let n be a positive integer greater than two. We keep the previous conventions and notations and denote any $\mathbb{Q}HS$ Seifert fibered 3-manifolds M with its normalized Seifert invariant, by : $M = M(-b_0, b_1/a_1, b_2/a_2, \dots, b_n/a_n)$.

Let $\mathcal{SF}_1$ be the set of all Seifert fibered 3-manifolds for which $b_0 = 1$ and which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry.

We denote by $\mathcal{Q}_n$ the set :

$$\mathcal{Q}_n = \mathcal{S}_n \cap \mathcal{SF}_1.$$

Then $\mathcal{Q}_n$ is the set of non-integral $\mathbb{Q}HS$ Seifert fibered 3-manifolds M with n exceptional fibers, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry and $M = M(-1, b_1/a_1, b_2/a_2, \dots, b_n/a_n)$.

This section is devoted to prove the following result, which clearly implies Main Theorem 2.

Theorem 7.1. *Let n be a positive integer greater than two. For each n :*

(i) *There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_n$ which admit a taut analytic foliation; and*

(ii) *There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_n$ which do not admit a taut $\mathcal{C}^2$-foliation.*

(iii) *There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_3$ which do not admit a taut $\mathcal{C}^0$-foliation.*

Proof. The proof of Theorem 7.1 is an immediate consequence of the two following lemmata. Let n be a positive integer greater than two. Let $\mathcal{M}(n)$ be the family of Seifert fibered 3-manifolds M with n exceptional fibers such that $M = M(-1, \dfrac{1}{2}, \dfrac{b_2}{a_2}, \dfrac{b_3}{a_3}, \dots, \dfrac{b_n}{a_n})$ and the exceptional slopes are ordered in the following way : $\dfrac{1}{2} > \dfrac{b_2}{a_2} \geq \dfrac{b_3}{a_3} \geq \cdots \geq \dfrac{b_n}{a_n}$.

Lemma 7.2. *Let n be a positive integer greater than two. We consider the following families of infinite Seifert fibered 3-manifolds.*

$$\mathcal{M}_1(n) = \left\{ M \in \mathcal{M}(n),\ \textit{with}\ \frac{b_2}{a_2} = \frac{3}{5},\ n > 3 \right\};$$
$$\mathcal{M}_1(3) = \left\{ M \in \mathcal{M}(3),\ \textit{with}\ \frac{b_2}{a_2} = \frac{3}{5},\ \textit{and}\ a_3 \geq 4 \right\};$$
$$\mathcal{M}_2(n) = \left\{ M \in \mathcal{M}(n),\ \textit{with}\ \frac{b_2}{a_2} = \frac{2}{5},\ \frac{b_3}{a_3} > \frac{1}{5}, n > 3 \right\};$$
$$\mathcal{M}_2(3) = \left\{ M \in \mathcal{M}(3),\ \textit{with}\ \frac{b_2}{a_2} = \frac{2}{5},\ \frac{b_3}{a_3} > \frac{1}{5},\ \textit{and}\ a_3 \geq 4 \right\}.$$

If $M \in \mathcal{M}_1(n) \cup \mathcal{M}_2(n)$, then $M \in \mathcal{Q}_n$. In particular, M is a non-integral homology 3-sphere, which admits the $\widetilde{SL_2}(\mathbb{R})$-geometry, and M does not admit a taut $\mathcal{C}^2$-foliation.

Furthermore, if $M \in \mathcal{M}_1(3) \cup \mathcal{M}_2(3)$, then $M \in \mathcal{Q}_3$, and M does not admit a taut $\mathcal{C}^0$-folitation.

Proof. First, considering Lemma 5.1, we may check easily that if $M \in \mathcal{M}_1(n) \cup \mathcal{M}_2(n)$ then M is a $\mathbb{Q}HS$ but not a $\mathbb{Z}HS$.

Indeed if $M \in \mathcal{M}_1(n)$, then $e(M) > -1 + 1/2 + 3/5$.

If $M \in \mathcal{M}_2$(n), then $e(M) > -1 + 1/2 + 2/5 + 1/5$.

In both cases, $e(M) > 1/10$, so $e(M) \neq 0$; hence, M is a $\mathbb{Q}HS$.

On the other hand, if $e(M) = \dfrac{\varepsilon}{a_1 a_2 \dots a_n}$ (where $\varepsilon = \pm 1$) then $e(M) < \dfrac{1}{10 a_3}$; which is a contradiction. Then, M is not a $\mathbb{Z}HS$.

Now, we check that they all have the $\widetilde{SL_2}(\mathbb{R})$-geometry.

If $n \geq 4$, then it is a direct consequence of Proposition 5.4.

If $n = 3$, that follows from $\displaystyle\sum_{i=1}^{3} \frac{1}{a_i} < 1$ (here, we need that $a_3 \geq 4$).

In conclusion, $\mathcal{M}_1(n) \cup \mathcal{M}_2(n) \subset \mathcal{Q}_n$ (for $n \geq 3$).

Finally, we check that they do not admit a taut $\mathcal{C}^2$-foliation.

If $M \in \mathcal{M}_1(n)$, Corollary 4.5 (1) implies that M cannot admit a taut $\mathcal{C}^2$-foliation.

If $M \in \mathcal{M}_2(n)$, then $\dfrac{b_2}{a_2}$ and $\dfrac{b_3}{a_3}$ both are greater than 1/5; therefore (iii) implies that $m \leq 4$. Thus, $\alpha \in \{1, 2, 3\}$. In all cases, (i) or (ii) cannot be satisfied.

Furthermore, by Theorem 1.3, if $M \in \big(\mathcal{M}_1(3) \cup \mathcal{M}_2(3)\big)$ and M admits a taut $\mathcal{C}^0$-folitation, then the foliation can be isotoped to be horizontal; which is impossible for M in $\mathcal{M}_1(3) \cup \mathcal{M}_2(3)$. □

Lemma 7.3. *Let n be a positive integer greater than two. Let $\mathcal{M}_3$ and $\mathcal{M}_4(n)$ be the two following families of infinite Seifert fibered 3-manifolds.*

$$\mathcal{M}_3 = \left\{ M\big(-1, \frac{1}{2}, \frac{2}{5}, \frac{k}{7k+1}\big) \in \mathcal{M}(3),\ k \in \mathbb{Z},\ k \geq 1 \right\};$$
$$\mathcal{M}_4(n) = \left\{ M\big(-1, \frac{1}{2}, \frac{2}{5}, \frac{1}{10}, \frac{b_4}{10b_4+1}, \dots, \frac{b_n}{10b_n+1}\big) \in \mathcal{M}(n),\ n > 3 \right\}.$$

If $M \in \mathcal{M}_3 \cup \mathcal{M}_4(n)$, then $M \in \mathcal{Q}_n$ and is a non-integral Seifert fibered 3-manifold, which admits the $\widetilde{SL}_2(\mathbb{R})$-geometry and a taut analytic foliation.

Proof. First, considering Lemma 5.1, we can check that if $M \in \mathcal{M}_3 \cup \mathcal{M}_4$, then M is a $\mathbb{Q}HS$ but not a $\mathbb{Z}HS$.

Indeed, if $M \in \mathcal{M}_3$, then $e(M) > -1 + 1/2 + 2/5 + 1/8$,
i.e. $e(M) > 1/40$; so $e(M) \neq 0$ and M is a $\mathbb{Q}HS$.
If $e(M) = \dfrac{\varepsilon}{a_1 a_2 a_3}$ (where $\varepsilon = \pm 1$) then $e(M) < 1/70$;

which is not possible so M is not a $\mathbb{Z}HS$.

Similarly, if $M \in \mathcal{M}_4$, then $e(M) > -1 + 1/2 + 2/5 + 1/10 + 1/11$,
i.e. $e(M) > 1/11$; so $e(M) \neq 0$ and M is a $\mathbb{Q}HS$.
If $e(M) = \dfrac{\varepsilon}{a_1 a_2 \dots a_n}$ then $e(M) < 1/100$, which is not possible so M is not a $\mathbb{Z}HS$.

Now, we check that they all admit the $\widetilde{SL}_2(\mathbb{R})$-geometry.
If $n \geq 4$, then it is a direct consequence of Proposition 5.4.
If $n = 3$, that follows from $\displaystyle\sum_{i=1}^{n} \frac{1}{a_i} < 1$.

Finally, if we choose $\alpha = 3$ and $m = 7$ then (m, α) trivially satisfies Property $(*)$; which implies that they all admit a taut analytic foliation (by Theorems 4.1 and 4.3). $\square$

End of proof of Theorem 7.1 $\square$

8. Proof of Main Theorem 1

This section is almost entirely devoted to the proof of Proposition 8.1, which implies Main Theorem 1, as it will be shown bellow.

We may recall here (see Section 2) that if M is a Seifert fibered 3-manifold, then $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$, where b_0 is a positive integer and $0 < b_i < a_i$ for all $i \in \{1, \dots, n\}$. Note that n has to be greater than 2 (otherwise M cannot be a $\mathbb{Z}HS$ but $\mathbb{S}^3$).

If M is also a $\mathbb{Z}HS$, then two rational coefficients cannot be the same, see Remark 5.2; therefore we may re-order them so that $b_1/a_1 > b_2/a_2 > \cdots > b_n/a_n$.

Thus, two positive integers m and α satisfy Property $(*)$ (for these rational coefficients) if and only if :

$$\boxed{\alpha < m \text{ and } (i) \text{ to } (iii) \text{ of Property } (*) \text{ are all satisfied.}}$$

Proposition 8.1. *Let n be a positive integer and M be a $\mathbb{Z}HS$ Seifert fibered 3-manifold, which is neither homeomorphic to $\mathbb{S}^3$ nor to $\mathcal{P}$.*
We assume that $M = M(-1, b_1/a_1, \dots, b_n/a_n)$, where :
- $0 < b_i < a_i$ for all $i \in \{1, \dots, n\}$, and;
- $b_1/a_1 > b_2/a_2 > \cdots > b_n/a_n$.
Then there exist two positive integers m and α which satisfy Property $(*)$.

Proof of Main Theorem 1

First of all, if M is either homeomorphic to $\mathbb{S}^3$ or to the Poincaré sphere $\mathcal{P}$, then we may recall that M cannot admit a taut foliation.

We assume that M is neither homeomorphic to $\mathbb{S}^3$ nor to the Poincaré sphere $\mathcal{P}$. We want to show that M always admit a taut analytic foliation.

Let $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$, where b_0 is a positive integer and $0 < b_i < a_i$ for all $i \in \{1, \ldots, n\}$.

First, we may note that Corollary 4.2 claims that if $b_0 \in \{2, \ldots, n-2\}$ then M admit a horizontal analytic foliation, which are taut $\mathcal{C}^2$-foliation. Then, we assume for the following that $b_0 \notin \{2, \ldots, n-2\}$.

On the other hand, since M is a $\mathbb{Z}HS$, Lemma 5.1 (1) implies that

$$b_0 = \sum_{i=1}^{n} \frac{b_i}{a_i} + \frac{\varepsilon}{a_1 a_2 \ldots a_n}, \text{ where } \varepsilon \in \{-1, +1\}.$$

Then, the property $0 < b_i/a_i < 1$ for all $i \in \{1, \ldots, n\}$, implies that $0 < b_0 < n$. By the fiber-preserving homeomorphism Φ (see Section 2) we may assume that $b_0 = 1$. Hence, Proposition 8.1 implies that there exists a pair of positive integers (m, α) which satisfy Property $(*)$. This implies that M admits a horizontal foliation (Theorem 4.3) then a taut analytic foliation (Lemma 2.2 and Theorem 4.1); which ends the proof of Main Theorem 1. □

The remaining of the paper is entirely devoted to the proof of Proposition 8.1.

Schedule of the proof of Proposition 8.1
The proof of Proposition 8.1 is organized in four steps, as follows.

Step 1 : If Proposition 8.1 is true for $n = 3$ then it is true for all $n \geq 3$.
Step 2 : Considering $n = 3$ gives common notations and results for the following.
Step 3 : We show Proposition 8.1 for $n = 3$ and $\epsilon = -1$.
Step 4 : We show Proposition 8.1 for $n = 3$ and $\epsilon = 1$.

Before starting the proof, we fix some notations and conventions for all the following of the paper.

Notations - Conventions
We keep the previous notations.

Let $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$ be a $\mathbb{Z}HS$ Seifert fibered 3-manifold, where $0 < b_i < a_i$ for all $i \in \{1, \ldots, n\}$.

By Lemma 5.1, M is a $\mathbb{Z}HS$ if and only if :

$$(E1) \qquad \sum_{i=1}^{n} \frac{b_i}{a_i} = 1 + \frac{\epsilon}{a_1.a_2.\ldots.a_n}, \text{ where } \epsilon \in \{-1, 1\}$$

Let $\hat{a}_i$ (for $i \in \{1,\dots,n\}$), $\alpha_1, \alpha_2, a'_3, b'_3$ be the following positive rational numbers. Note that all are positive integers but α_1, α_2, which are rational numbers.

$$\boxed{\begin{array}{ll} \alpha_1 = 1 - \dfrac{b_1}{a_1} & \alpha_2 = \dfrac{b_2}{a_2} \\[2ex] \hat{a}_i = \dfrac{a_3 \dots a_n}{a_i} & \forall i \in \{3,\dots,n\} \\[2ex] b'_3 = \displaystyle\sum_{i=3}^{n} b_i \hat{a}_i & a'_3 = a_3 \dots a_n \end{array}}$$

Thus,

$$\frac{b'_3}{a'_3} = \sum_{i=3}^{n} \frac{b_i}{a_i}.$$

Now, we fix the following inequalities by denoting them from (1) to (6). The former three are trivially always true. The last three are true when $n = 3$, see Claim 8.2; they concern Steps 2 to 4.

$$\boxed{\begin{array}{ll} \multicolumn{2}{c}{(1) \quad \dfrac{b_1}{a_1} > \dfrac{b_2}{a_2} > \dots > \dfrac{b_n}{a_n}} \\[2ex] (2) \quad \dfrac{b_1}{a_1} \geq \dfrac{1}{2} & (3) \quad \alpha_1 \leq \dfrac{b_1}{a_1} \end{array}}$$

$$\boxed{\begin{array}{lll} \text{When } n = 3: & & \\[1ex] (4) \quad \dfrac{b_2}{a_2} < \dfrac{1}{2} & (5) \quad \dfrac{b_3}{a_3} < \dfrac{1}{4} & (6) \quad \alpha_2 > \alpha_1 - \alpha_2 \end{array}}$$

(1) up to reordering;
(2) by Corollary 4.4, which implies (3);
(4) to (6), by Claim 8.2.

When $n = 3$, $a'_3 = a_3$ and $b'_3 = b_3$, and (E_1) is equivalent to :

$$\boxed{(E2) \qquad \frac{b_3}{a_3} = \alpha_1 - \alpha_2 + \frac{\epsilon}{a_1 a_2 a_3}, \ \textit{where } \epsilon \in \{-1, 1\}.}$$

Claim 8.2. *If $n = 3$ then $\dfrac{b_2}{a_2} < \dfrac{1}{2}$, $\dfrac{b_3}{a_3} < \dfrac{1}{4}$ and $\alpha_2 > \alpha_1 - \alpha_2$.*

Proof. Since $b_1/a_1 \geq 1/2$, there exists a non-negative integer r_1 such that

$$2b_1 = a_1 + r_1.$$

If $b_2/a_2 + b_3/a_3 < 1/2$ then (1) implies (4) and (5). So, we may suppose that $b_2/a_2 + b_3/a_3 \geq 1/2$. Hence, there exists a non-negative integer r such that :

$$2(b_2 a_3 + a_2 b_3) = a_2 a_3 + r.$$

Then $(E2)$ implies that $\dfrac{r_1}{2a_1} + \dfrac{r}{2a_2 a_3} = \dfrac{\epsilon}{a_1 a_2 a_3}$, so $r_1 a_2 a_3 + r a_1 = 2\epsilon$.

Therefore, $r_1 = 0,\ r = 1,\ a_1 = 2$, and $\epsilon = +1$.
Thus, $b_1/a_1 = 1/2$ and (1) implies (4) and $a_3b_2 > a_2b_3$.
Then $2(b_2a_3 + a_2b_3) = a_2a_3 + r$ implies $1 + a_2a_3 > 4a_2b_3$ and so $a_2a_3 \geq 4a_2b_3$, which is equivalent to $1/4 \geq b_3/a_3$.
Since $a_1 = 2$ and the a_i's are pairwise relatively prime, $1/4 > b_3/a_3$ which proves (5).

By $(E2)$, $\alpha_1 - \alpha_2 = \dfrac{b_3}{a_3} - \dfrac{\epsilon}{a_1a_2a_3}$.

On the other hand, (1) implies : $b_2a_3 \geq b_3a_2 + 1$ (since they are positive integers).

Therefore, $\alpha_2 = \dfrac{b_2}{a_2} \geq \dfrac{b_3}{a_3} + \dfrac{1}{a_2a_3} > \dfrac{b_3}{a_3} - \dfrac{\epsilon}{a_1a_2a_3}$ which implies (6). □

8.1. **Step 1 : From $n = 3$ to $n > 3$.**
We suppose that Proposition 8.1 is satisfied for $n = 3$. Now, we assume that $n \geq 4$ and $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$ is a $\mathbb{Z}HS$. We want to show that Property $(*)$ is satisfied for the rational coefficients of the Seifert invariant of M.
Let $M' = M(-1, b_1/a_1, b_2/a_2, b'_3/a'_3)$. Note that (E_1) is satisfied because M is a $\mathbb{Z}HS$; therefore M' is also a $\mathbb{Z}HS$, by the definition of b'_3/a'_3.

We separate the proof according to either $\dfrac{b'_3}{a'_3} < \dfrac{b_2}{a_2}$, or $\dfrac{b_2}{a_2} < \dfrac{b'_3}{a'_3}$.

Note that $\dfrac{b_2}{a_2} \neq \dfrac{b'_3}{a'_3}$ because the a_i's are pairwise relatively prime.

Case 1 : $\dfrac{b'_3}{a'_3} < \dfrac{b_2}{a_2}$.

First, we check that $M' \ncong \mathcal{P}$. Indeed, otherwise $\dfrac{b'_3}{a'_3} = \dfrac{1}{5}$, so $a'_3 = a_3 \ldots a_n = 5$, with $n \geq 4$; a contradiction. Then, there exist positive integers m and α such that $\alpha < m$ and :

(i) $\dfrac{b_1}{a_1} < \dfrac{m-\alpha}{m}$;

(ii) $\dfrac{b_2}{a_2} < \dfrac{\alpha}{m}$; and

(iii) $\dfrac{b'_3}{a'_3} < \dfrac{1}{m}$.

By definition, $\dfrac{b_i}{a_i} < \dfrac{b'_3}{a'_3}$ for $i \in \{3, 4, \ldots, n\}$, then the same positive integers m and α satisfy Property $(*)$ for the rational coefficients $\dfrac{b_i}{a_i}$ (for $i \in \{1, \ldots, n\}$).

Case 2 : $\dfrac{b_2}{a_2} < \dfrac{b'_3}{a'_3}$.

We repeat the same argument.
Similarly, $M' \ncong \mathcal{P}$; otherwise $\dfrac{b'_3}{a'_3} = \dfrac{1}{3}$, so $a_3 \ldots a_n = 3$, with $n \geq 4$; a contradiction.
Then, there exist positive integers m and α such that $\alpha < m$ and :

(i) $\frac{b_1}{a_1} < \frac{m-\alpha}{m}$;

(ii) $\frac{b'_3}{a'_3} < \frac{\alpha}{m}$; and

(iii) $\frac{b_2}{a_2} < \frac{1}{m}$.

Since $b_1/a_1 > b_2/a_2 > \cdots > b_n/a_n$, we obtain that $\frac{b_i}{a_i} < \frac{1}{m}$ for $i \in \{2, 3, \ldots, n\}$, which implies that m and α can be chosen so that they satisfy Property $(*)$ for the rational coefficients $\frac{b_i}{a_i}$ (for $i \in \{1, \ldots, n\}$).

8.2. **Step 2 : General results for $n = 3$.**
First, note that if m and α are positive integers such that $\alpha < m$, which satisfy Property $(*)$ then, by definition of α_1 and α_2 : (i) and (ii) of Property $(*)$ are respectively equivalent to (I) and (II) bellow.

$$\begin{cases} (I) & \alpha < m\alpha_1 \\ (II) & m\alpha_2 < \alpha \end{cases}$$

Let

$$\boxed{\begin{gathered} a = a_1 a_2 \text{ and } b = a - b_1 a_2 - b_2 a_1; \\ \text{then } \frac{b}{a} = \alpha_1 - \alpha_2. \end{gathered}}$$

Let $[.]$ denote the integral value,

i.e. $[x]$ is the integer k such that $k \leq x < k+1$, for all real x.
Let $N = [a/b]$, hence

$$\boxed{N = \left[\frac{1}{\alpha_1 - \alpha_2}\right].}$$

Lemma 8.3. *Recall that α and m are integers. The two following properties are satisfied.*

(i) *$N \geq 4$;*
(ii) *If $N\alpha_1 - 1 \leq \alpha \leq N\alpha_1$ and $N - 1 \leq m$, then $0 < \alpha < m$.*

Proof. Proof of (i). By $(E2)$ and (5), $\alpha_1 - \alpha_2 < \frac{1}{4} - \frac{\varepsilon}{aa_3}$, i.e. $4b < a - \frac{4\varepsilon}{a_3}$.

Note that (5) implies that $a_3 \geq 5$ $(b_3 \geq 1)$.

Then (since a and b are positive integers) $4b \leq a$. So $N = \left[\frac{a}{b}\right] \geq 4$.

Proof of (ii). Let α and m such that $N\alpha_1 - 1 \leq \alpha \leq N\alpha_1$ and $N - 1 \leq m$.
Now, we can check that $0 < \alpha < m$.
The fact that $\alpha < m$ is trivial because $\alpha_1 \leq 1/2$.
Let's check that $\alpha \geq 1$.
First, note that if $b = 1$ then $N\alpha_1 - 1 = a\frac{(a_1 - b_1)}{a_1} - 1 = a_2(a_1 - b_1) - 1 = b_2 a_1 > 1$.
Then, we assume $b > 1$.

We proceed by contradiction. Assume $\alpha = 0$, then $N\alpha_1 \leq 1$.

$N\alpha_1 \leq 1 \Leftrightarrow \alpha_1 \leq \dfrac{1}{N}$, which is $\dfrac{1}{[a/b]}$.

Hence, (E_2) implies $\dfrac{b_2}{a_2} + \dfrac{b_3}{a_3} \leq \dfrac{1}{[a/b]} + \dfrac{\epsilon}{a_1 a_2 a_3}$.

Since $\dfrac{b_3}{a_3} < \dfrac{b_2}{a_2}$, $\dfrac{b_3}{a_3} \leq \dfrac{1}{2[a/b]} + \dfrac{\epsilon}{2a_1 a_2 a_3}$ and so $2b_3[a/b] \leq a_3 + \dfrac{\epsilon[a/b]}{a}$.

Now, $b > 1$ implies $\dfrac{[a/b]}{a} < 1$ hence $2b_3[a/b] \leq a_3$.

Furthermore $[a/b] > a/b - 1 \Rightarrow \dfrac{a}{b} - 1 < \dfrac{a_3}{2b_3}$ and so : $ab_3 - bb_3 < \dfrac{a_3 b}{2}$.

Then $ab_3 - \dfrac{a_3 b}{2} < bb_3$.

Finally, note that $(E_2) \Leftrightarrow ab_3 - a_3 b = \epsilon$, i.e. $ab_3 - \dfrac{a_3 b}{2} = \epsilon + \dfrac{a_3 b}{2}$.

Hence $\epsilon + \dfrac{a_3 b}{2} < bb_3 \Leftrightarrow \dfrac{b_3}{a_3} > \dfrac{1}{2} + \dfrac{\epsilon}{ba_3}$.

By (5) $\varepsilon = -1$ and $\dfrac{1}{4} > \dfrac{1}{2} + \dfrac{-1}{ba_3}$, i.e. $\dfrac{1}{ba_3} > \dfrac{1}{4}$, so $ba_3 < 4$.

This is a contradiction because (5) implies that $a_3 \geq 5$ and $b \geq 2$. ☐

Lemma 8.4. *Let $r = N\alpha_1 - [N\alpha_1]$, $r' = a/b - [a/b]$ and $r'' = a\alpha_1/b - [a\alpha_1/b]$.*
If $N\alpha_1 \in \mathbb{N}$, let $(\alpha, m) = (N\alpha_1 - 1, N - 1)$.
If $N\alpha_1 \notin \mathbb{N}$ and $r'\alpha_2 \leq r'' < \alpha_1 r'$, let $(\alpha, m) = ([N\alpha_1], N)$.
Otherwise, let $(\alpha, m) = ([N\alpha_1], N - 1)$.

Then the integers m and α are positive integers which satisfy (I) and (II) and $\alpha < m$.

The proof of this lemma is the main part of Step 3, but does not depend on $\varepsilon = \pm 1$. The fact that $0 < \alpha < m$ is an immediate consequence of Lemma 8.3.

8.3. **Step 3 : $n = 3$ and $\epsilon = -1$.**
Let us consider *Property* $(**)$ bellow :

$$(**)\begin{cases} (I) & \alpha < m\alpha_1 \\ (II) & m\alpha_2 < \alpha \\ (III) & \dfrac{b}{a} < \dfrac{1}{m} \end{cases}$$

By $(E2)$: $\epsilon = -1 \Rightarrow \dfrac{b_3}{a_3} < \dfrac{b}{a}$, then Property $(**)$ implies trivially Property $(*)$, i.e. if there exist positive integers m and α, such that $\alpha < m$ which satisfy Property $(**)$, then they satisfy Property $(*)$.
We will separate the cases where $N\alpha_1 \in \mathbb{N}$ or $N\alpha_1 \notin \mathbb{N}$. If $N\alpha_1 \notin \mathbb{N}$, let

$$\boxed{r = N\alpha_1 - [N\alpha_1] \qquad r' = a/b - [a/b] \qquad r'' = a\alpha_1/b - [a\alpha_1/b]}$$

Claim 8.5. $N\alpha_1 = \dfrac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r'$.

Proof. By definition of r', $N\alpha_1 = [a/b]\alpha_1 = (a/b - r')\alpha_1$.
Then $N\alpha_1 = \dfrac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r'$. $\square$

Claim 8.6. $N\alpha_1 = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] + r'' - \alpha_1 r'$.

Proof. By Claim 8.5 $\dfrac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r' = N\alpha_1$.

Moreover, $\dfrac{\alpha_1}{\alpha_1 - \alpha_2} = \dfrac{a\alpha_1}{b} = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] + r''$, by definition of r''. $\square$

Claim 8.7. *If* $[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] - 1$ *then* $r'' = r + \alpha_1 r' - 1$.

Proof. First, we may note that $N\alpha_1 = [N\alpha_1] + r$, by definition of r.

Assume that $[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] - 1$.

By Claim 8.5, $\dfrac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r' = N\alpha_1 = [N\alpha_1] + r = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] - 1 + r$.

Hence $\dfrac{\alpha_1}{\alpha_1 - \alpha_2} = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] + \alpha_1 r' + r - 1$.

So $r'' = r + \alpha_1 r' - 1$, by definition of r''. $\square$

We want to find positive integers m and α such that $\alpha < m$, and which satisfy Property $(**)$. First, we consider separately the case $b = 1$.

Lemma 8.8. *If* $b = 1$ *then* $m = a - 1$ *and* $\alpha = a_1 b_2$ *satisfy property* $(*)$ *and* $0 < \alpha < m$.

Proof. Assume that $b = 1$ and let $m = a - 1$ and $\alpha = a_1 b_2$. First, we can check that $0 < \alpha < m$ because $a_1 b_2 \leq a_1(a_2 - 1) < a_1 a_2 - 1$. Now, we want to check successively (I) to (III).

$(I) \Leftrightarrow \alpha < m\alpha_1$.

$m\alpha_1 = (a_1 a_2 - 1)\dfrac{a_1 - b_1}{a_1} > a_2(a_1 - b_1) - 1$.

Since $b = 1$, $a_2(a_1 - b_1) - 1 = a_1 b_2 < m\alpha_1$.

$(II) \Leftrightarrow m\alpha_2 < \alpha$.

$m\alpha_2 = (a_1 a_2 - 1)\dfrac{b_2}{a_2} = a_1 b_2 - \dfrac{b_2}{a_2} < \alpha$.

$(III) \Leftrightarrow \dfrac{b}{a} < \dfrac{1}{m}$.

Since $b = 1$ and $m = a - 1$, (III) is direct. $\square$

In the following of this section, we assume that $b \neq 1$. We distinguish the three following cases.

CASE A : $N\alpha_1 \in \mathbb{N}$. Then $(\alpha, m) = (N\alpha_1 - 1, N - 1)$.
CASE B : $N\alpha_1 \notin \mathbb{N}$ and $r'\alpha_2 > r''$ or $r'' \geq \alpha_1 r'$. Then $(\alpha, m) = ([N\alpha_1], N)$.
CASE C : $N\alpha_1 \notin \mathbb{N}$ and $r'\alpha_2 \leq r'' < \alpha_1 r'$. Then $(\alpha, m) = ([N\alpha_1], N - 1)$.

First, we prove (III) of Property $(**)$. Then Lemma 8.4 concludes this step. Note that, for $\varepsilon = 1$, Lemmata 8.8 and 8.4 imply that (I) to (III) are true, but (III) does not imply (iii).

Furthermore, we may note that $b \neq 1$ if and only if $[\frac{a}{b}] < \frac{a}{b}$ because a and b are positive coprime integers (since a_1 and a_2 are so).

Lemma 8.9. *We assume that $b \neq 1$. If the integers α and m are chosen as in Lemma 8.4 (according to Cases A, B or C) then $\frac{b}{a} < \frac{1}{m}$.*

Proof. Let α and m be integers as in Cases A, B and C successively.

Assume that Case A or Case C is satisfied.

Then $m = N-1$. Therefore (III) is trivial, because $m = N-1 = [a/b]-1 < a/b$, so $1/m > b/a$.

Assume now that Case B is satisfied.

Then $(III) \Leftrightarrow b/a < 1/N \Leftrightarrow N < a/b$, which is satisfied because $N = [a/b]$ and $b \neq 1$. $\square$

Proof of Lemma 8.4

We may recall that the proof does not depend on $\varepsilon = \pm 1$.

We only have to show that the considered integers in Cases A, B and C satisfy (I) and (II). We may recall that $0 < \alpha < m$ by Lemma 8.3.

$$\boxed{\text{CASE } A: \ N\alpha_1 \in \mathbb{N},\ (\alpha, m) = (N\alpha_1 - 1, N - 1)}$$

$(I) \Leftrightarrow \alpha < m\alpha_1$.

So, $(I) \Leftrightarrow N\alpha_1 - 1 < (N-1)\alpha_1 \Leftrightarrow \alpha_1 < 1$ which is true because $0 < \frac{b_1}{a_1} < 1$.

$(II) \Leftrightarrow m\alpha_2 < \alpha$.
$(II) \Leftrightarrow (N-1)\alpha_2 < N\alpha_1 - 1 \Leftrightarrow 1 - \alpha_2 < N(\alpha_1 - \alpha_2)$.

Therefore, $(II) \Leftrightarrow \dfrac{1}{\alpha_1 - \alpha_2} - N < \dfrac{\alpha_2}{\alpha_1 - \alpha_2}$.

But recall that $N = [\dfrac{1}{\alpha_1 - \alpha_2}]$, hence $\dfrac{1}{\alpha_1 - \alpha_2} - N < 1$. Thus, (II) follows from Claim 8.2 (6).

$$\boxed{\text{CASE } B: \ r'' \geq \alpha_1 r' \text{ OR } r'' < r'\alpha_2,\ (\alpha, m) = ([N\alpha_1], N)}$$

$(I) \Leftrightarrow \alpha < m\alpha_1$.
(I) is trivially satisfied : $(I) \Leftrightarrow [N\alpha_1] < N\alpha_1$.

$(II) \Leftrightarrow m\alpha_2 < \alpha$.

$(II) \Leftrightarrow N\alpha_2 < [N\alpha_1] \Leftrightarrow N\alpha_2 < N\alpha_1 - r$.

Then $(II) \Leftrightarrow r < N(\alpha_1 - \alpha_2) \Leftrightarrow r < (a/b - r')(\alpha_1 - \alpha_2)$, by definition of r'.

Recall that $b/a = \alpha_1 - \alpha_2$, so

$(II) \Leftrightarrow r < 1 - r'(\alpha_1 - \alpha_2) \Leftrightarrow r + r'(\alpha_1 - \alpha_2) < 1$.

We want to prove that $r + r'(\alpha_1 - \alpha_2) < 1$.

Assume first, that $r'' \geq \alpha_1 r'$.
Then Claim 8.6 implies that : $[N\alpha_1] = [\frac{\alpha_1}{\alpha_1 - \alpha_2}]$.
By Claim 8.5 $\frac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r' = [N\alpha_1] + r$, then $[N\alpha_1] = \frac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r' - r$.
Thus $\frac{\alpha_1}{\alpha_1 - \alpha_2} = [\frac{\alpha_1}{\alpha_1 - \alpha_2}] + \alpha_1 r' + r$; so $r'' = \alpha_1 r' + r < 1$.
Now, we can see that : $r + r'(\alpha_1 - \alpha_2) < r + \alpha_1 r' < 1$, which proves (II).

Now, we may assume that $r'' < r'\alpha_2$.
By the previous work, we may assume that $r'' < \alpha_1 r'$.
Then Claim 8.6 implies that $[N\alpha_1] = [\frac{\alpha_1}{\alpha_1 - \alpha_2}] - 1$.
Therefore, Claim 8.7 implies that $r'' = r + \alpha_1 r' - 1$.
Recall that we want to show that $r + r'(\alpha_1 - \alpha_2) < 1$.
Since $r'' < r'\alpha_2$, we obtain :
$r + r'(\alpha_1 - \alpha_2) = r + \alpha_1 r' - r'\alpha_2 < r + \alpha_1 r' - r''$.
Here, $r + \alpha_1 r' - r'' = 1$, which gives the required inequality.

$\boxed{\text{Case } C : r'' < \alpha_1 r' \text{ and } r'' \geq r'\alpha_2,\ (\alpha, m) = ([N\alpha_1], N-1)}$

$(I) \Leftrightarrow \alpha < m\alpha_1$.
$(I) \Leftrightarrow [N\alpha_1] < (N-1)\alpha_1 \Leftrightarrow \alpha_1 < r$.

Since $r'' < \alpha_1 r'$, by Claim 8.6 : $[N\alpha_1] = [\frac{\alpha_1}{\alpha_1 - \alpha_2}] - 1$.

Then, by Claim 8.7 : $r'' = r + \alpha_1 r' - 1$.

Thus $(I) \Leftrightarrow \alpha_1 < r'' - \alpha_1 r' + 1 \Leftrightarrow \alpha_1 r' - r'' < 1 - \alpha_1$.

Hence, $(I) \Leftrightarrow \alpha_1 r' - r'' < \frac{b_1}{a_1}$, because $1 - \alpha_1 = \frac{b_1}{a_1}$.

On the other hand, $\alpha_1 r' - r'' < \alpha_1 - r''$ and $\alpha_1 - r'' \leq \alpha_1 \leq \frac{b_1}{a_1}$ by (3).
Therefore (I) is satisfied.

$(II) \Leftrightarrow m\alpha_2 < \alpha$.
$(II) \Leftrightarrow (N-1)\alpha_2 < [N\alpha_1]$.

By Claim 8.6 and the definition of r'', and since $r'' < \alpha_1 r'$:

$[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1-\alpha_2}] - 1 = \dfrac{\alpha_1}{\alpha_1-\alpha_2} - r'' - 1.$

Moreover, by the definition of r' : $N\alpha_2 = \dfrac{\alpha_2}{\alpha_1-\alpha_2} - \alpha_2 r'$.

Hence $(II) \Leftrightarrow N\alpha_2 < [N\alpha_1] + \alpha_2 \Leftrightarrow \dfrac{\alpha_2}{\alpha_1-\alpha_2} - \alpha_2 r' < \dfrac{\alpha_1}{\alpha_1-\alpha_2} - r'' - 1 + \alpha_2$.

Therefore, $(II) \Leftrightarrow r'' - r'\alpha_2 < \alpha_2$.

On the other hand, $\alpha_2 > \alpha_1 - \alpha_2$ by (6) and $r'\alpha_2 \leq r'' < \alpha_1 r'$.

Then, $r'' - r'\alpha_2 < r'(\alpha_1-\alpha_2) < r'\alpha_2 < \alpha_2$, which proves that (II) is satisfied.

Proof of Lemma 8.4

□

In conclusion, Lemma 8.8 solve the case $b = 1$. If $b \neq 1$, then for the α and m chosen as in Lemma 8.3, we get that $0 < \alpha < m$ and Lemmata 8.4 together with 8.9 show that they satisfy $(I), (II)$ and (III). Therefore, Property $(*)$ is satisfied for $n = 3$ and $\varepsilon = -1$.

8.4. **Step 4 : $n = 3$ and $\epsilon = 1$.**
Recall that $a = a_1a_2$ and $b = a - b_1a_2 - b_2a_1$.

We assume that $\epsilon = 1$ then $(E2)$ gives :

$$(E3) \qquad \frac{b_3}{a_3} = \frac{b}{a} + \frac{1}{a_1a_2a_3}$$

so :

$$(7) \qquad ab_3 - ba_3 = 1.$$

Then (Bezout relation) there exists a unique pair of positive coprime integers (u, v) such that :

$$(8)\begin{cases} au - bv = 1; \\ 0 < u \leq b & \text{and} \\ 0 < v \leq a \end{cases}$$

Now, (7) implies that there exists $p \in \mathbb{N}$ such that

$$\begin{cases} b_3 = u + bp & \text{and} \\ a_3 = v + ap \end{cases}$$

Moreover, for all $p \in \mathbb{N}$, we have :

$$(9) \qquad \frac{u}{v} \geq \frac{u+bp}{v+ap} > \frac{u+b(p+1)}{v+a(p+1)} > \frac{b}{a}$$

We want to find positive integers α and m such that $\alpha < m$ and satisfy Property $(*)$. We consider separately the three following cases.

Case I : $u \neq 1$.
Case II : $u = 1$ and $b = 1$.
Case III : $u = 1$ and $b \neq 1$.

$\boxed{\text{Case I} : u \neq 1}$

We will choose the integers α and m as in Lemma 8.4, so $m \in \{N-1, N\}$. By (9), if $\frac{u}{v} < \frac{1}{m}$, then (iii) of Property $(*)$ is satisfied. Therefore, Lemma 8.4 and the following lemma conclude Case I.

Lemma 8.10. *If $N-1 \leq m \leq N$ and $u \neq 1$, then $\frac{u}{v} < \frac{1}{m}$.*

Proof. Assume that $N-1 \leq m \leq N$ and $u \neq 1$. First, note that $b \neq 1$, because $0 < u \leq b$ implies that if $b = 1$ then $u = 1$.

We make the proof by contradiction. So, we suppose that $\frac{u}{v} \geq \frac{1}{m}$, and we look for a contradiction. Note that $v = um$ cannot happen, because u, v are coprime integers, and u and m are at least 2, by Lemma 8.3. Thus $\frac{u}{v} > \frac{1}{m}$.

Moreover, by Lemma 8.9 : $\frac{b}{a} < \frac{1}{m}$. Then

$$\frac{b}{a} < \frac{1}{m} < \frac{u}{v}$$

By (8) :

$$\frac{a}{b} - \frac{v}{u} = \frac{1}{ub}$$

we obtain

$$0 < m - \frac{v}{u} < \frac{a}{b} - \frac{v}{u} = \frac{1}{ub} < 1$$

which implies that

$$\left[\frac{v}{u}\right] = m - 1$$

Now, let

$$r' = \frac{a}{b} - \left[\frac{a}{b}\right] < 1, \text{ and } \rho = \frac{v}{u} - \left[\frac{v}{u}\right] < 1$$

We consider separately the cases $m = N$ and $m = N-1$.

First, assume that $m = N = \left[\frac{a}{b}\right]$.

Then $\left[\frac{v}{u}\right] = \left[\frac{a}{b}\right] - 1 \Leftrightarrow \frac{a}{b} - r' - 1 = \frac{v}{u} - \rho$;

hence $\frac{1}{ub} = 1 + r' - \rho \Rightarrow 1 + r' - \rho < \frac{1}{b}$ because $u \neq 1$.

Thus $r' < \frac{1}{b}$ because $\rho < 1$.

Nevertheless, $r' = \frac{a}{b} - \left[\frac{a}{b}\right]$, a and b are coprime, and $a > b$.

Hence $a = bk + l$, where $k \in \mathbb{N}^*$, and $1 \leq l \leq b - 1$;

so r' can be written $r' = k + \frac{l}{b} - \left[k + \frac{l}{b}\right] = \frac{l}{b} \Rightarrow r' \geq \frac{1}{b}$; which is a contradiction.

Now, assume that $m = N - 1 = \left[\frac{a}{b}\right] - 1$.

Then $\left[\frac{v}{u}\right] = \left[\frac{a}{b}\right] - 2 \Leftrightarrow \frac{a}{b} - r' - 2 = \frac{v}{u} - \rho$;

hence $\dfrac{1}{ub} = 2 + r' - \rho$.

This implies that $\dfrac{1}{ub} > 1$ because r' and ρ lies in $[0, 1[$.

On the other hand, $\dfrac{1}{ub} \leq \dfrac{1}{4}$, because $b \geq 2$ and $u \geq 2$.

These are in a contradiction.

(Lemma 8.10)

□

$\boxed{\text{CASE II : } u = 1 \text{ AND } b = 1}$

We assume that $u = b = 1$. Then $au - bv = 1$ gives $v = a - 1$.

We consider separately the cases where $\dfrac{b_1}{a_1} > \dfrac{1}{2}$ or $\dfrac{b_1}{a_1} = \dfrac{1}{2}$.

First, assume that $\dfrac{b_1}{a_1} > \dfrac{1}{2}$.

Let $m = a - 2$, and $\alpha = a_2(a_1 - b_1) - 1 = a - a_2 b_1 - 1 = b_2 a_1$ (because $b = a - a_2 b_1 - a_1 b_2 = 1$. Then $0 < \alpha < m$.

We want to check $(I), (II)$ and (iii).

$(I) \Leftrightarrow \alpha < m\alpha_1 \Leftrightarrow a_2(a_1 - b_1) - 1 < (a_1 - b_1)a_2 - 2\alpha_1 \Leftrightarrow \dfrac{1}{2} < \dfrac{b_1}{a_1}$ (which is satisfied here).

$(II) \Leftrightarrow m\alpha_2 < \alpha \Leftrightarrow (a-2)\alpha_2 < a_2(a_1 - b_1) - 1 \Leftrightarrow 1 - 2\alpha_2 < a_2(a_1 - b_1) - a\alpha_2$; which is satisfied because $a_2(a_1 - b_1) - a\alpha_2 = a_2(a_1 - b_1) - a_1 b_2 = b = 1$.

By (9), (iii) is satisfied if $\dfrac{u}{v} < \dfrac{1}{m}$; which is true because $\dfrac{u}{v} = \dfrac{1}{a-1}$ and $\dfrac{1}{m} = \dfrac{1}{a-2}$.

Now, assume that $\dfrac{b_1}{a_1} = \dfrac{1}{2}$.

Then $a_1 = 2$ and $b_1 = 1$. Since $1 = b = a_2(a_1 - b_1) - a_1 b_2$, $a_2 = 1 + 2b_2$. So :

$$\frac{b_2}{a_2} = \frac{b_2}{1 + 2b_2}$$

and by $(E2)$: $\dfrac{b_3}{a_3} = \dfrac{1}{2} - \dfrac{b_2}{1 + 2b_2} + \dfrac{1}{2(2b_2 + 1)a_3}$.

Thus, $\dfrac{b_3}{a_3} = \dfrac{(2b_2 + 1)a_3 - 2b_2 a_3 + 1}{2(2b_2 + 1)a_3}$, i.e.

$$\frac{b_3}{a_3} = \frac{a_3 + 1}{2(2b_2 + 1)a_3}$$

We consider separately the cases $b_2 = 1$ and $b_2 > 1$.
Assume first $b_2 = 1$.

Then $a_2 = 3$ so $\dfrac{b_3}{a_3} = \dfrac{a_3 + 1}{6a_3}$.

Therefore, we can check easily that $\alpha = 2$ and $m = 5$ satisfy Property $(*)$.

(i) $\quad \dfrac{b_1}{a_1} = \dfrac{1}{2} < \dfrac{m-\alpha}{m}$, which is $\dfrac{3}{5}$;

(ii) $\quad \dfrac{b_2}{a_2} = \dfrac{1}{3} < \dfrac{\alpha}{m}$, which is $\dfrac{2}{5}$; and

(iii) $\quad \dfrac{b_3}{a_3} = \dfrac{a_3+1}{6a_3} < \dfrac{1}{m}$ (which is $\dfrac{1}{5}$) if and only if $a_3 > 5$.

By (5) $a_3 \geq 5$, but if $a_3 = 5$ then $M \cong \mathcal{P}$, so $a_3 > 5$.

Now, we assume that $b_2 \geq 2$.

Let $\alpha = 2b_2 - 1$ and $m = 4b_2 - 1$.

Since $b_2 \geq 2 :\ 0 < \alpha < m$. We want to check $(i), (ii)$ and (iii).

(i) $\quad \dfrac{b_1}{a_1} = \dfrac{1}{2} < \dfrac{m-\alpha}{m}$, which is $\dfrac{2b_2}{4b_2-1}$; so (i) is satisfied.

(ii) $\quad \dfrac{b_2}{a_2} = \dfrac{b_2}{2b_2+1} < \dfrac{\alpha}{m}$, which is $\dfrac{2b_2-1}{4b_2-1}$;

and $\dfrac{b_2}{2b_2+1} < \dfrac{2b_2-1}{4b_2-1}$ if and only if $4b_2^2 - b_2 < 4b_2^2 - 1$, i.e. $b_2 > 1$;

so (ii) is satisfied.

(iii) $\quad \dfrac{b_3}{a_3} = \dfrac{a_3+1}{2(2b_2+1)a_3} < \dfrac{1}{m}$, which is $\dfrac{1}{4b_2-1}$.

Then, (iii) is satisfied if and only if :
$(a_3+1)(4b_2-1) < (4b_2+2)a_3$ i.e. $4b_2 < 3a_3 + 1$.

Since $\dfrac{b_3}{a_3} = \dfrac{a_3+1}{2(2b_2+1)a_3}$, $b_3 = \dfrac{a_3+1}{2(2b_2+1)} \geq 1$ (because b_3 is a positive integer).

So $a_3 + 1 \geq 4b_2 + 2$; thus (iii) is satisfied.

Case III : $u = 1$ and $b \neq 1$

We assume $u = 1$ and $b \geq 2$. Then $a - bv = 1$ by (8).

Claim 8.11. *If* $\dfrac{b_2}{a_2} < \dfrac{1}{v}$ *then* $m = v$ *and* $\alpha = 1$ *satisfy Property* $(*)$.

Proof. Assume that $\dfrac{b_2}{a_2} < \dfrac{1}{v}$. To prove that $m = v$ and $\alpha = 1$ satisfy Property $(*)$, it remains to prove that $\dfrac{b_1}{a_1} < \dfrac{v-1}{v}$. Indeed, (II) and (iii) are trivially satisfied because $\dfrac{b_3}{a_3} < \dfrac{b_2}{a_2} < \dfrac{1}{v}$.

By (8) : $1 + \dfrac{1}{av} = \dfrac{b_1}{a_1} + \dfrac{b_2}{a_2} + \dfrac{1}{v}$.

But $\dfrac{b_2}{a_2} > \dfrac{1}{av}$, otherwise $b_2 < \dfrac{1}{a_1 v}$; which is impossible.

Therefore, $\dfrac{b_1}{a_1} = 1 + \dfrac{1}{av} - \dfrac{b_2}{a_2} - \dfrac{1}{v} < 1 - \dfrac{1}{v}$, so $\dfrac{b_1}{a_1} < \dfrac{v-1}{v}$. □

Hence, in the following, we assume that $\dfrac{b_2}{a_2} > \dfrac{1}{v}$ (note that the equality is impossible because the integers are coprime).

Let α be the integer such that $(v-1)\alpha_1 - 1 \leq \alpha < (v-1)\alpha_1$, and $m = min(v-1, M)$, where M is the positive integer such that $\dfrac{\alpha}{\alpha_2} - 1 \leq M < \dfrac{\alpha}{\alpha_2}$. Then :

$$\boxed{\begin{gathered} \alpha = (v-1)\alpha_1 - r, \text{ where } 0 < r \leq 1 \\ M = \frac{\alpha}{\alpha_2} - r', \text{ where } 0 < r' \leq 1 \\ \text{and } m = \min(M, v-1). \end{gathered}}$$

First, we will check that $m > \alpha > 0$, then we will show that the integers m and α satisfy Property $(*)$.

Claim 8.12. *The integers m and α satisfy : $1 \leq \alpha < m$*

Proof. First, we check that $\alpha \geq 1$, where $\alpha = (v-1)\alpha_1 - r$, $0 < r \leq 1$.
We show that $(v-1)\alpha_1 > 1$, then $\alpha > 0$. Since $\alpha \in \mathbb{N}$, $\alpha \geq 1$.

By (8) : $\alpha_1 = \dfrac{b_2}{a_2} + \dfrac{1}{v} - \dfrac{1}{a_1 a_2 v}$.

Since $\dfrac{b_2}{a_2} > \dfrac{1}{v}$, $\alpha_1 > \dfrac{2}{v} - \dfrac{1}{a_1 a_2 v}$, i.e. $\alpha_1 > \dfrac{2a_1a_2 - 1}{a_1 a_2 v}$.

Therefore $v > \dfrac{2a_1a_2 - 1}{a_1 a_2 \alpha_1}$, so $(v-1)\alpha_1 > \dfrac{a_1a_2(2-\alpha_1) - 1}{a_1 a_2}$.

Finally, recall that $1 - \alpha_1 = \dfrac{b_1}{a_1}$.

Thus, $(v-1)\alpha_1 > \dfrac{a_1 a_2(1 + \frac{b_1}{a_1}) - 1}{a_1 a_2}$, i.e. $(v-1)\alpha_1 > \dfrac{a_1a_2 + a_2 b_1 - 1}{a_1 a_2}$.

Since $a_2 \geq 3$, $(v-1)\alpha_1 > \dfrac{a_1 a_2 + 2}{a_1 a_2} > 1$.

Now, we check that $m > \alpha$.
If $m = v - 1$, this is trivial.
So, we may assume that $m = \dfrac{\alpha}{\alpha_2} - r'$, where $0 < r' \leq 1$.

Therefore, $m = \alpha(\dfrac{1}{\alpha_2} - \dfrac{r'}{\alpha})$.

Since $\alpha \geq 1$, $\dfrac{r'}{\alpha} \leq r' \leq 1$, so $m \geq \alpha(\dfrac{1}{\alpha_2} - 1)$.

Finally, (4) implies that $\alpha_2 < \dfrac{1}{2}$ and so that $m > \alpha$. □

To show that α and m satisfy Property $(*)$, we need the following claim.

Claim 8.13. $\dfrac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1<1-\dfrac{1}{a}$.

Proof. We first consider that $\dfrac{b_1}{a_1}=\dfrac{1}{2}$.

Then $\alpha_1=\dfrac{1}{2}$ and $a=2a_2$; so $1-\dfrac{1}{a}=\dfrac{2a_2-1}{2a_2}$.

On the other hand, $\dfrac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1=\dfrac{3}{2}-2\alpha_2=\dfrac{3a_2-4b_2}{2a_2}$.

Then, here : $\dfrac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1<1-\dfrac{1}{a}$ if and only if

$$a_2-4b_2<-1.$$

Now, (6) implies $\alpha_2>\dfrac{\alpha_1}{2}$, i.e. $4b_2>a_2$, so $a_2-4b_2\leq -1$.

We are going to show that $a_2\neq 4b_2-1$ by contradiction.
First, note that since $b_1/a_1=1/2$, $a=2a_2$ and $b=a_2-2b_2\neq 1$.

On the other hand, since $a-bv=1$, $v=\dfrac{a-1}{b}=\dfrac{2a_2-1}{a_2-2b_2}$.

If $a_2=4b_2-1$, then $v=\dfrac{8b_2-3}{2b_2-1}$.

Now, $v=4+\dfrac{1}{2b_2-1}\in\mathbb{N}$ implies that $b_2=1$, $v=5$ and $a_2=3$. Then $b=3-2=1$; which is a contradiction.

Therefore, $a_2<4b_2-1$; which is the required inequality.

Now, we assume that $\dfrac{b_1}{a_1}>\dfrac{1}{2}$, so $2b_1-a_1>0$.

Then $a_1-b_1<a_1b_2(2b_1-a_1)$,

so $(a_1-b_1)+a_1^2b_2<2a_1b_1b_2$,

and $(a_1-b_1)-2a_1b_1b_2+2a_1^2b_2<a_1^2b_2$. Therefore :

$$(\star)\qquad (a_1-b_1)(1+2a_1b_2)<a_1^2b_2.$$

On the other hand, (6) implies that $2\alpha_2>\alpha_1$, i.e. $2a_1b_2>a_2(a_1-b_1)$.
Hence, $2a_1b_2(a_1-b_1)+(a_1-b_1)>a_2(a_1-b_1)^2+(a_1-b_1)$, i.e.

$$(2a_1b_2+1)(a_1-b_1)>a_2(a_1-b_1)^2+(a_1-b_1).$$

Therefore, by the inequality $(\star)$:

$$a_2(a_1-b_1)^2+(a_1-b_1)<a_1^2b_2.$$

So $\dfrac{a_1-b_1}{a_1^2a_2}<\dfrac{a_1^2b_2-a_2(a_1-b_1)^2}{a_1^2a_2}$; i.e. $\dfrac{\alpha_1}{a}<\alpha_2-\alpha_1^2$.

Thus, we obtain $\dfrac{1}{a}<\dfrac{\alpha_2}{\alpha_1}-\alpha_1$, which gives the required inequality. □

Now, we will show successively that α and m satisfy (iii), (II) and (I) of Property $(*)$.

- *α and m satisfy (iii) : $\frac{b_3}{a_3} < \frac{1}{m}$.*

This is trivially satisfied because by (9), $\frac{b_3}{a_3} \leq \frac{1}{v}$, and $m \leq v-1$.

- *α and m satisfy (II) : $m\alpha_2 < \alpha$.*

Since $m \leq M$, $m\alpha_2 \leq \alpha - r'\alpha_2 < \alpha$ (because $r' > 0$) then (α, m) trivially satisfies (II).

- *α and m satisfy (I) : $\alpha < m\alpha_1$.*

Since $r > 0$, $(v-1)\alpha_1 - r < (v-1)\alpha_1$. Hence, $\alpha < m\alpha_1$ if $m = v-1$. Thus, we may assume that $m = M \leq v-2$.

So, we want to show that $(v-1)\alpha_1 - r < (\frac{\alpha}{\alpha_2} - r')\alpha_1$. Now :

$$\begin{aligned}
(v-1)\alpha_1 - r < (\frac{\alpha}{\alpha_2} - r')\alpha_1 \quad &\Leftrightarrow \quad v - \frac{r}{\alpha_1} < \frac{(v-1)\alpha_1 - r}{\alpha_2} - r' + 1 \\
&\Leftrightarrow \quad v\alpha_1\alpha_2 - r\alpha_2 < v\alpha_1^2 - \alpha_1^2 - r\alpha_1 - r'\alpha_1\alpha_2 + \alpha_1\alpha_2 \\
&\Leftrightarrow \quad r(\alpha_1 - \alpha_2) + r'\alpha_1\alpha_2 + \alpha_1(\alpha_1 - \alpha_2) < v\alpha_1(\alpha_1 - \alpha_2) \\
&\Leftrightarrow \quad v(\alpha_1 - \alpha_2) > \alpha_1 - \alpha_2 + r\frac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2
\end{aligned}$$

Recall that $\frac{b}{a} = \alpha_1 - \alpha_2$ and $a - bv = 1$, so :

$$v(\alpha_1 - \alpha_2) = \frac{vb}{a} = \frac{a-1}{a} = 1 - \frac{1}{a}.$$

Therefore, α and m satisfy (I) if and only if

$$1 - \frac{1}{a} > \alpha_1 - \alpha_2 + r\frac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2.$$

Since r and r' both lie in $]0, 1]$:

$$\alpha_1 - \alpha_2 + r\frac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2 < \alpha_1 - \alpha_2 + \frac{\alpha_1 - \alpha_2}{\alpha_1} + \alpha_2,$$

i.e. $\alpha_1 - \alpha_2 + r\frac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2 < \alpha_1 + \frac{\alpha_1 - \alpha_2}{\alpha_1} < 1 - \frac{1}{a}$, by Claim 8.13.

Hence, α and m satisfy (I), which ends the proof of Proposition 8.1.

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